POSITIVE SOLUTIONS TO SCHRÖDINGER EQUATIONS AND GEOMETRIC APPLICATIONS

OVIDIU MUNTEANU, FELIX SCHULZE AND JIAPING WANG

Abstract. A variant of Li-Tam theory, which associates to each end of a complete Riemannian manifold a positive solution of a given Schrödinger equation on the manifold, is developed. It is demonstrated that such positive solutions must be of polynomial growth of fixed order under a suitable scaling invariant Sobolev inequality. Consequently, a finiteness result for the number of ends follows. In the case when the Sobolev inequality is of particular type, the finiteness result is proven directly. As an application, an estimate on the number of ends for shrinking gradient Ricci solitons and submanifolds of Euclidean space is obtained.

1. Introduction

Recall that a complete manifold (M, g) is a gradient shrinking Ricci soliton if there exists a function f on M such that the Ricci curvature of M and the hessian of f satisfy the equation

$$\mathrm{Ric} + \mathrm{Hess}(f) = \frac{1}{2}\, g.$$

As self-similar solutions to the Ricci flow, gradient shrinking Ricci solitons arise naturally from singularity analysis of the Ricci flow. Indeed, according to [40, 31, 13, 6], the blow-ups around a type-I singularity point always converge to nontrivial gradient shrinking Ricci solitons. It is thus a central issue in the study of the Ricci flow to understand and classify gradient shrinking Ricci solitons. While the issue has been successfully resolved for dimension 2 and 3 (see [18, 35, 31, 33, 3]), it remains open for dimension 4, though recent work [29, 30, 12] has shed some light on it. Presently, there is very limited information available concerning general gradient shrinking Ricci solitons in higher dimensions.

The potential f and the scalar curvature S are related through the following equation [18]

$$|\nabla f|^2 + S = f \tag{1.1}$$

with f normalized by adding a suitable constant. By [8], $S > 0$ unless (M, g) is the Euclidean space. Moreover, according to [4, 17], there exists a point $p \in M$ and constants $c_1(n)$, $c_2(n)$ depending only on the dimension n of M such that

$$\frac{1}{4}r^2(x) - c_1(n)r(x) - c_2(n) \leq f(x) \leq \frac{1}{4}r^2(x) + c_1(n)r(x) + c_2(n) \tag{1.2}$$

for all $x \in M$, where $r(x) = d(p, x)$ is the distance from p to x, and the volume $\mathrm{V}_p(r)$ of the geodesic ball $B_p(r)$ centered at p of radius r satisfies

$$\mathrm{V}_p(r) \leq c(n)\, r^n. \tag{1.3}$$

Perelman's entropy is given by

$$\mu(g) = \ln\left(\frac{1}{(4\pi)^{\frac{n}{2}}}\int_M e^{-f}\right). \tag{1.4}$$

Set

$$\alpha = \limsup_{R\to\infty} \frac{1}{\mathrm{V}_p(R)}\int_{B_p(R)} \left(S\,r^2\right)^{\frac{n-1}{2}}. \tag{1.5}$$

We have the following result.

Theorem 1.1. *Let (M,g) be a gradient shrinking Ricci soliton with $\alpha<\infty$. Then the number of ends of M is bounded from above by $\Gamma(n,\alpha,\mu(g))$, a constant depending only on dimension n, $\mu(g)$ and α.*

A gradient shrinking Ricci soliton M is called asymptotically conical if there exists a closed Riemannian manifold (Σ, g_Σ) and diffeomorphism

$$\Phi : (R,\infty)\times\Sigma \to M\setminus\Omega$$

such that $\lambda^{-2}\,\rho_\lambda^*\,\Phi^*\,g$ converges in C^∞_{loc} as $\lambda\to\infty$ to the cone metric $dr^2+r^2\,g_\Sigma$ on $[R,\infty)\times\Sigma$, where Ω is a compact smooth domain of M. Clearly, an asymptotically conical shrinking Ricci soliton must satisfy $\alpha<\infty$.

Recall that an end of a complete manifold M with respect to a compact smooth domain $\Omega\subset M$ is simply an unbounded component of $M\setminus\Omega$. The number of ends $e(M)$ of M is the maximal number obtained over all such Ω. The novelty of Theorem 1.1 is that only the scalar curvature integral information at infinity is needed. Another feature is that the exponent of S in the definition of α is $\frac{n-1}{2}$, not the commonly seen $\frac{n}{2}$ in analysis. We emphasize that the estimate here is explicit. That M has finitely many ends follows readily by assuming the scalar curvature of M is bounded. Indeed, as observed in [14], (1.1) and (1.2) imply that $|\nabla f|\geq 1$ outside a compact subset of M and hence M must have finite topological type. We mention here that in [28] it was shown that any complete shrinking Kähler Ricci soliton must have one end. The proof uses Li-Tam's theory and a fact special to the Kähler situation that the gradient vector ∇f is real holomorphic.

For shrinking gradient Ricci solitons of dimension $n\geq 3$, by Li-Wang [26], the following Sobolev inequality holds.

$$\left(\int_M \phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C(n)\,e^{-\frac{2\mu(g)}{n}}\int_M\left(|\nabla\phi|^2+S\phi^2\right)$$

for $\phi\in C_0^\infty(M)$. So Theorem 1.1 is a consequence of the following general result.

Theorem 1.2. *Let (M,g) be a complete Riemannian manifold of dimension $n\geq 3$ satisfying the Sobolev inequality*

$$\left(\int_M \phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq A\int_M\left(|\nabla\phi|^2+\sigma\phi^2\right)$$

for any $\phi\in C_0^\infty(M)$, where $A>0$ is a constant and $\sigma\geq 0$ a continuous function. Suppose

$$\alpha = \limsup_{R\to\infty}\frac{1}{\mathrm{V}_p(R)}\int_{B_p(R)}\left(r^2\sigma\right)^{\frac{n-1}{2}}<\infty$$

and

$$V_\infty = \limsup_{R\to\infty} \frac{\mathrm{V}_p(R)}{R^n} < \infty.$$

Then the number of ends of M *is bounded above by a constant* Γ *depending only on* n, A, α *and* V_∞.

The well known Michael-Simon inequality [2, 27] for submanifolds in the Euclidean space $\mathbb{R}^N$ states that

$$\left(\int_M |\phi|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le C(n) \int_M \left(|\nabla\phi| + |H|\,|\phi|\right) \tag{1.6}$$

for any $\phi \in C_0^\infty(M)$, where H is the mean curvature vector of M. In fact, this inequality holds for submanifolds in Cartan-Hadamard manifolds as well [15]. These inequalities are particularly useful in studying minimal submanifolds. We refer to [7, 32, 5, 36] and the references therein for some of the results. It is easy to see that

$$\left(\int_M \phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C(n) \int_M \left(|\nabla\phi|^2 + |H|^2\phi^2\right)$$

holds for $n \ge 3$. As a corollary of Theorem 1.2, we have the following result.

Corollary 1.3. *Let* M^n *be a complete submanifold of* $\mathbb{R}^N$ *with* $n \ge 2$. *Suppose*

$$\alpha = \limsup_{R\to\infty} \frac{1}{\mathrm{V}_p(R)} \int_{B_p(R)} (r\,|H|)^{n-1} < \infty$$

and

$$V_\infty = \limsup_{R\to\infty} \frac{\mathrm{V}_p(R)}{R^n} < \infty.$$

Then the number of ends of M *is bounded above by a constant* Γ *depending only on the dimension* n, α *and* V_∞.

Strictly speaking, for the case of dimension $n = 2$, the conclusion does not follow directly from Theorem 1.2. Rather, it follows by a slight modification of its proof. Our proof of Theorem 1.2 is very much motivated by the work of Topping [42, 43], where the diameter of a compact manifold M satisfying the Sobolev inequality is estimated in terms of the constant A together with the integral $\int_M \sigma^{\frac{n-1}{2}}$. The argument there is adapted to show that for each large R, the volume of $E \cap B_p(R)$ satisfies $\mathrm{V}(E \cap B_p(R)) \ge c\,R^n$ for some constant c for at least one half of the ends E of M. Note that for different R the choice of such set of ends E may be different. Nonetheless, the desired estimate of the number of ends follows as the total volume of the ball $B_p(R)$ is at most of $2V_\infty R^n$. We emphasize that the argument strongly depends on the fact that the Sobolev exponent is of $\frac{n}{n-2}$ with n being the dimension of the manifold. For a Sobolev inequality with general exponent $\mu > 1$ of the form

$$\left(\int_M \phi^{2\mu}\right)^{\frac{1}{\mu}} \le A \int_M \left(|\nabla\phi|^2 + \sigma\phi^2\right)$$

for $\phi \in C_0^\infty(M)$, we instead develop a different approach of using positive solutions to a Schrödinger equation to estimate the number of ends.

More specifically, the approach relies on a variant of Li-Tam theory. In [23], to each end E of M, they associate a harmonic function f_E on M. The resulting harmonic functions are linearly independent. So the question of bounding the number of ends $e(M)$ is reduced to estimating the dimension of the space spanned by those functions. The theory was successfully applied to show that $e(M)$ is necessarily finite when the Ricci curvature of M is nonnegative outside a compact set. We shall refer to [22] for more applications of this theory. Here, we develop a variant of their theory by considering instead the Schrödinger operator

$$L = \Delta - \sigma$$

with σ being a nonnegative but not identically zero smooth function on M.

Theorem 1.4. *Let (M, g) be a complete manifold and $E_1, E_2, \cdots, E_l$ the ends of M with respect to a geodesic ball $B_p(r_0)$ of M with $l \geq 2$. Then for each end E_i, there exists a positive solution u_i to the equation $\Delta u_i = \sigma u_i$ on M satisfying $0 < u_i \leq 1$ on $M \backslash E_i$ and*

$$\sup_M u_i = \limsup_{x \to E_i(\infty)} u_i(x) > 1.$$

Moreover, the functions $u_1, \cdots, u_l$ are linearly independent.

One nice feature here is that all the functions u_i are positive, while in the case of harmonic functions f_E is positive if and only if M is nonparabolic, that is, it admits a positive Green's function. With this result in hand, we set out to bound the dimension of the space $\mathcal{F}$ spanned by the functions $u_1, \cdots, u_l$. The work of [10, 11, 21] on the dimension of spaces of harmonic functions with polynomial growth inspires us to consider the mean value property for positive subsolutions to L. More precisely, assume that M admits a proper Lipschitz function $\rho > 0$ satisfying

$$\frac{1}{2} \leq |\nabla \rho| \leq 1 \text{ and } \Delta \rho \leq \frac{m}{\rho}, \tag{1.7}$$

in the weak sense for $\rho \geq R_0$, a sufficiently large constant and some constant $m > 0$.

Denote the sublevel and level sets of ρ by

$$\begin{aligned} D(r) &= \{x \in M : \rho(x) < r\} \\ \Sigma(r) &= \{x \in M : \rho(x) = r\}. \end{aligned}$$

To simplify notation, we let $\mathrm{V}(r) = \mathrm{Vol}(D(r))$ and $\mathrm{A}(r) = \mathrm{Area}(\Sigma(r))$.

Definition 1.5. *A manifold (M, g) has the mean value property $(\mathcal{M})$ if there exist constants $A_0 > 0$ and $\nu > 1$ such that for any $0 < \theta \leq 1$ and $R \geq 4R_0$,*

$$\sup_{\Sigma(R)} u \leq \frac{A_0}{\theta^{2\nu}} \frac{1}{\mathrm{V}((1+\theta)R)} \int_{D((1+\theta)R) \backslash D(R_0)} u \tag{1.8}$$

holds true for any function $u > 0$ satisfying $\Delta u \geq \sigma u$ on $D(2R) \backslash D(R_0)$.

With this definition at hand, we can now state our main estimate on positive solutions to the Schrödinger equation $Lu = 0$. For $q \geq 1$, define the quantity

$$\alpha = \limsup_{R \to \infty} \left(R^{2q} \fint_{\Sigma(R)} \sigma^q \right)^{\frac{1}{q}}, \tag{1.9}$$

where

$$\fint_{\Sigma(R)} \sigma^q = \frac{1}{\mathrm{A}(R)} \int_{\Sigma(R)} \sigma^q.$$

Theorem 1.6. *Assume that (M, g) admits a proper function ρ satisfying (1.7) and has the mean value property $(\mathcal{M})$. For a polynomially growing positive solution u of $\Delta u = \sigma u$ on $M \backslash D(R_0)$, if $\alpha < \infty$ for some $q > \nu - \frac{1}{2}$, then there exists a constant $\Gamma(m, A_0, \nu, \alpha) > 0$ such that*

$$u \leq \Lambda\left(\rho^{\Gamma} + 1\right) \quad \text{on } M \backslash D(R_0),$$

where $\Lambda > 0$ is a constant depending on u. In the critical case $q = \nu - \frac{1}{2}$, the same conclusion holds true with $\Gamma = \Gamma(m, A_0, \nu)$ provided that $\alpha \leq \alpha_0(m, A_0, \nu)$, a sufficiently small positive constant.

This result is reminiscent of Agmon type estimates in [1, 24, 25], where a positive subsolution u to L is shown to decay at a certain rate if it does not grow too fast, provided that a Poincaré type inequality holds on M. Whether a positive solution u to $Lu = 0$, under the assumptions in Theorem 1.6, is automatically of polynomial growth is unclear at this point. But we do confirm this is the case under a pointwise assumption on $\sigma > 0$ that

$$\sup_{M}\left(\rho^2 \sigma\right) < \infty. \tag{1.10}$$

If we let

$$L^d(M) = \left\{v : \Delta v = \sigma v, |v| \leq c\,\rho^d \ \text{ on } M\right\},$$

the space of polynomial growth solutions of degree at most d, then an argument verbatim to [21] immediately gives the following estimate of the dimension.

Lemma 1.7. *Assume that (M, g) admits a proper function ρ satisfying (1.7) and has mean value property $(\mathcal{M})$. Then $\dim L^d(M) \leq \Gamma(m, A_0, \nu, d)$.*

Summarizing, we have the following conclusion, where $\mathcal{P}$ is the space spanned by all positive solutions to the equation $\Delta u = \sigma u$ on M.

Theorem 1.8. *Assume that (M, g) admits a proper function ρ satisfying (1.7) and has mean value property $(\mathcal{M})$. Suppose that σ decays quadratically. Then $\dim \mathcal{P} \leq \Gamma(m, A_0, \nu, \alpha)$ provided that $\alpha < \infty$ for some $q > \nu - \frac{1}{2}$. In the critical case $q = \nu - \frac{1}{2}$, the same conclusion holds for some $\Gamma(m, A_0, \nu)$ when $\alpha \leq \alpha_0(m, A_0, \nu)$, a sufficiently small positive constant. Consequently, the number of ends $e(M)$ of M satisfies the same estimate as well.*

It is well known that the mean value property $(\mathcal{M})$ is implied by the following scaling invariant Sobolev inequality via a Moser iteration argument with the number ν determined by the Sobolev exponent μ through the equation

$$\frac{1}{\mu} + \frac{1}{\nu} = 1.$$

Definition 1.9. *(M, g) is said to satisfy the Sobolev inequality $(\mathcal{S})$ if there exist constants $\mu > 1$ and $A > 0$ such that*

$$\left(\fint_{D(R)} \phi^{2\mu}\right)^{\frac{1}{\mu}} \leq AR^2 \fint_{D(R)} \left(|\nabla \phi|^2 + \sigma \phi^2\right) \tag{1.11}$$

for $\phi \in C_0^{\infty}(D(R))$ and $R \geq R_0$.

We have denoted with

$$\fint_{D(R)} u = \frac{1}{\mathrm{V}(R)} \int_{D(R)} u$$

for any integrable function u on $D(R)$. Consequently, Theorem 1.8 continues to hold if one replaces the mean value property $(\mathcal{M})$ by the Sobolev inequality $(\mathcal{S})$.

We also establish a version of Theorem 1.6 localized to an end.

For an end E of M, define

$$\alpha_E = \limsup_{R\to\infty} \left(\frac{R^{2q}}{\mathrm{A}\,(R)} \int_{\partial E(R)} \sigma^q \right)^{\frac{1}{q}},$$

where $E(R) = E \cap D(R)$ and $\partial E(R) = E \cap \Sigma(R)$.

Proposition 1.10. *Assume that (M, g) admits a proper function ρ satisfying (1.7) and that the Sobolev inequality $(\mathcal{S})$ holds. Suppose that σ decays quadratically along E. Then there exists $\Gamma\,(m, A, \mu, \alpha_E) > 0$ such that*

$$u \leq \Lambda \left(\rho^{\Gamma} + 1\right) \quad \text{on } E$$

for positive solutions u to $\Delta u = \sigma u$ on E, where $\Lambda > 0$ is a constant depending on u, provided that $\alpha_E < \infty$ for some $q > \nu - \frac{1}{2}$. In the case $q = \nu - \frac{1}{2}$, the same conclusion holds for some $\Gamma(m, A, \mu) > 0$ when $\alpha_E \leq \alpha_0(m, A, \mu)$, a sufficiently small positive constant.

Corresponding to an end E, let u_E be the positive solution of $\Delta u_E = \sigma u_E$ on M constructed in Theorem 1.4. Then $0 < u_E \leq 1$ on $M \setminus E$. Proposition 1.10 implies that such u_E must be of polynomial growth on M with the given growth order. With this in hand and in view of Lemma 1.7, for the case of critical $q = \nu - \frac{1}{2}$, one concludes that the number of ends with small α_E is bounded. For an asymptotically conical gradient shrinking Ricci soliton M, it is not difficult to show that at least one half of the ends have small α_E if the total number of ends is large. Obviously, Theorem 1.1 follows, at least for asymptotically conical shrinking Ricci solitons, from these facts as well.

Sobolev inequalities are prevalent in geometry. Other than the aforementioned ones for gradient shrinking Ricci solitons and submanifolds in the Euclidean spaces, for manifolds with Ricci curvature bounded from below by a constant $-K$, $K \geq 0$, according to [38], the Sobolev inequality (1.11) holds on any geodesic ball $B_p(R)$ with constant $A = e^{c(n)\left(1+\sqrt{K}R\right)}$ and $\sigma = \frac{1}{R^2}$. Finally, for a locally conformally flat manifold M, by [39], a suitable cover of M can be mapped conformally into $\mathbb{S}^n$ and satisfies a similar Sobolev inequality of gradient shrinking Ricci solitons.

For a comprehensive study of Sobolev inequalities on manifolds and their applications, we refer to [16, 37].

The paper is organized as follows. In Section 2, we present the proof of Theorem 1.2 and derive some of its consequences. In Section 3 we focus on the proof of Theorem 1.4. We then turn to estimates of positive solutions to $\Delta u = \sigma u$ in Section 4 and prove Theorem 1.6. The dimension estimate given in Lemma 1.7 is proved in Section 5. Section 6 is devoted to proving the fact that the mean value property $(\mathcal{M})$ follows from the Sobolev inequality $(\mathcal{S})$.

2. Sobolev inequality and ends

In this section, we prove Theorem 1.2 following the ideas in [42, 43]. To include the case $n = 2$, we consider more generally complete noncompact Riemannian manifolds (M, g) satisfying the Sobolev inequality

$$\left(\int_M |\phi|^{\frac{q\,n}{n-q}}\right)^{\frac{n-q}{n}} \leq A\int_M \left(|\nabla\phi|^q + \sigma|\phi|^q\right) \tag{2.1}$$

for some q with $1 \leq q \leq n-1$ and any $\phi \in C_0^\infty(M)$, where $A > 0$ is a constant and $\sigma \geq 0$ a continuous function. Define

$$\alpha = \limsup_{R\to\infty} \frac{1}{\mathrm{V}_p(R)} \int_{B_p(R)} (r^q\sigma)^{\frac{n-1}{q}} \tag{2.2}$$

and

$$V_\infty = \limsup_{R\to\infty} \frac{\mathrm{V}_p(R)}{R^n}, \tag{2.3}$$

where $p \in M$ is a fixed point, $r(x) = d(p, x)$ is the distance function to p, and $\mathrm{V}_p(R) = \mathrm{Vol}(B_p(R))$, the volume of the geodesic ball $B_p(R)$ centered at p of radius R.

We restate Theorem 1.2 below under this more general Sobolev inequality.

Theorem 2.1. *Let (M, g) be an n dimensional complete Riemannian manifold satisfying the Sobolev inequality (2.1). If both α of (2.2) and V_∞ of (2.3) are finite, then the number of ends of M is bounded from above by a constant Γ depending only on n, A, α and V_∞.*

Proof. For an end E of M we denote $E(R) = B_p(R) \cap E$. Assume that M has at least k ends with $k > 1$ large, to be specified later. We may take $R > 0$ large enough such that

$$B_p(2R)\backslash B_p(R) = \cup_{i=1}^k E_i(2R)\backslash E_i(R).$$

Moreover, we have from (2.3) that

$$\frac{\mathrm{V}_p(t)}{t^n} \leq 2\,V_\infty \tag{2.4}$$

for all $t \geq R$. Similarly, by (2.2) we have,

$$\sum_{i=1}^k \int_{E_i(3R)\backslash E_i(R)} (r^q\sigma)^{\frac{n-1}{q}} \leq 2\alpha\,\mathrm{V}_p(3R).$$

This implies that

$$\sum_{i=1}^k \int_{E_i(3R)\backslash E_i(R)} \sigma^{\frac{n-1}{q}} \leq C_0\,\frac{\mathrm{V}_p(3R)}{R^{n-1}}. \tag{2.5}$$

Here and below constants C_0, $C_1, \ldots$ depend only on n, A, α and V_∞.

We may assume that the ends $E_1, \cdots, E_k$ are labeled so that

$$\left\{\int_{E_i(3R)\backslash E_i(R)} \sigma^{\frac{n-1}{q}}\right\}_{i=1,\cdots,k}$$

is an increasing sequence. Then (2.5) implies that

$$\int_{E_i(3R)\setminus E_i(R)} \sigma^{\frac{n-1}{q}} \leq \frac{2C_0}{k}\frac{\mathrm{V}_p(3R)}{R^{n-1}} \tag{2.6}$$

for all $i=1,2,\cdots,\left[\frac{k}{2}\right]$.

For $i\in\left\{1,2,\cdots,\left[\frac{k}{2}\right]\right\}$, pick

$$z_i\in\partial E_i(2R). \tag{2.7}$$

By relabeling $E_1,..,E_{\left[\frac{k}{2}\right]}$ if necessary, we may assume that

$$\{\mathrm{V}_{z_i}(R)\}_{i=1,\cdots,\left[\frac{k}{2}\right]}$$

is increasing.

Assume by contradiction that

$$\mathrm{V}_{z_1}(R)\geq \frac{C_1}{k}\,R^n, \tag{2.8}$$

where $C_1=3^{n+2}\,V_\infty$. Since

$$B_{z_i}(R)\subset E_i(3R)\setminus E_i(R)$$

and $\{B_{z_i}(R)\}_{i=1}^{\left[\frac{k}{2}\right]}$ are disjoint in $B_p(3R)$, it follows from (2.8) that

$$\mathrm{V}_p(3R)\geq\sum_{i=1}^{\left[\frac{k}{2}\right]}\mathrm{V}_{z_i}(R)\geq\left[\frac{k}{2}\right]\frac{C_1}{k}\,R^n\geq\frac{C_1}{3}\,R^n=3V_\infty(3R)^n$$

as $C_1=3^{n+2}V_\infty$. This contradicts (2.4). In conclusion, (2.8) does not hold and

$$\mathrm{V}_{z_1}(R)<\frac{C_1}{k}\,R^n.$$

For convenience, from now on we simply write $E=E_1$ and $z=z_1$. Hence, we have $z\in\partial E(2R)$ and

$$\mathrm{V}_z(R)<\frac{C_1}{k}\,R^n. \tag{2.9}$$

By (2.6) we also have

$$\int_{E(3R)\setminus E(R)} \sigma^{\frac{n-1}{q}} \leq \frac{C_2}{k}\frac{\mathrm{V}_p(3R)}{R^{n-1}}. \tag{2.10}$$

Let $\gamma(t)$ be a minimizing geodesic from p to z with $0\leq t\leq 2R$. For $t\in\left[\frac{4}{3}\,R,\frac{5}{3}\,R\right]$ and $x=\gamma(t)$, since

$$d(x,z)\leq\frac{2}{3}\,R,$$

the triangle inequality implies

$$B_x\left(\frac{R}{3}\right)\subset B_z(R). \tag{2.11}$$

Consequently, (2.9) yields

$$\mathrm{V}_x\left(\frac{R}{3}\right)<\frac{C_1}{k}\,R^n \tag{2.12}$$

for all $x=\gamma(t)$ with $t\in\left[\frac{4}{3}\,R,\frac{5}{3}\,R\right]$.

Assume by contradiction that

$$\int_{B_x(r)} \sigma \leq \delta\, r^{\frac{q}{n-1}}\, \left(\mathrm{V}_x(r)\right)^{\frac{n-q-1}{n-1}} \tag{2.13}$$

for all $0 < r < \frac{R}{3}$, where $\delta > 0$ is small constant to be set later.

For $0 < r < \frac{R}{3}$ fixed, we apply the Sobolev inequality to cut-off function ϕ with support in $B_x(r)$ such that $\phi = 1$ on $B_x\left(\frac{r}{2}\right)$ and $|\nabla\phi| \leq \frac{2}{r}$. Then (2.1) implies that

$$\left(\mathrm{V}_x\left(\frac{r}{2}\right)\right)^{\frac{n-q}{n}} \leq A\left(\frac{2^q}{r^q}\,\mathrm{V}_x(r) + \int_{B_x(r)} \sigma\right). \tag{2.14}$$

Using (2.13) we obtain that

$$\left(\mathrm{V}_x\left(\frac{r}{2}\right)\right)^{\frac{n-q}{n}} \leq A\left(\frac{2^q}{r^q}\,\mathrm{V}_x(r) + \delta\, r^{\frac{q}{n-1}}\, \left(\mathrm{V}_x(r)\right)^{\frac{n-q-1}{n-1}}\right) \tag{2.15}$$

for any $0 < r < \frac{R}{3}$. Let us assume there exists $0 < r < \frac{R}{3}$ so that

$$\mathrm{V}_x(r) \leq \delta^{\frac{n-1}{q}}\, r^n. \tag{2.16}$$

Then by (2.15) we have

$$\left(\mathrm{V}_x\left(\frac{r}{2}\right)\right)^{\frac{n-q}{n}} \leq A(2^q+1)\,\delta^{\frac{n-1}{q}}\, r^{n-q}.$$

Hence,

$$\mathrm{V}_x\left(\frac{r}{2}\right) \leq 2^n\, \left(A(2^q+1)\right)^{\frac{n}{n-q}}\, \delta^{\frac{n-1}{n-q}}\, \delta^{\frac{n-1}{q}}\, \left(\frac{r}{2}\right)^n. \tag{2.17}$$

We now choose δ to be small enough so that

$$2^n\, \left(A(2^q+1)\right)^{\frac{n}{n-q}}\, \delta^{\frac{n-1}{n-q}} < 1.$$

Then (2.17) implies

$$\mathrm{V}_x\left(\frac{r}{2}\right) \leq \delta^{\frac{n-1}{q}}\, \left(\frac{r}{2}\right)^n. \tag{2.18}$$

In conclusion, assuming that (2.13) holds for any $0 < r < \frac{R}{3}$, we have shown that (2.16) implies (2.18).

By assuming k to be sufficiently large such that $\frac{3^n C_1}{k} \leq \delta^{\frac{n-1}{q}}$, (2.12) says that

$$\mathrm{V}_x\left(\frac{R}{3}\right) \leq \delta^{\frac{n-1}{q}}\, \left(\frac{R}{3}\right)^n,$$

that is, (2.16) holds for $r = \frac{R}{3}$. Applying (2.16) and (2.18) inductively, we conclude that

$$\mathrm{V}_x\left(\frac{R}{3\cdot 2^m}\right) \leq \delta^{\frac{n-1}{q}}\, \left(\frac{R}{3\cdot 2^m}\right)^n$$

for all $m \geq 0$. Letting $m \to \infty$ we reach a contradiction by further arranging δ to be sufficiently small such that $\delta^{\frac{n-1}{q}} < \omega_n$, the volume of the unit ball in the Euclidean space $\mathbb{R}^n$.

The contradiction implies that (2.13) does not hold. Therefore, for any $x = \gamma(t)$, $t \in \left[\frac{4}{3}\,R, \frac{5}{3}\,R\right]$, there exists $0 < r_x < \frac{R}{3}$ such that

$$\int_{B_x(r_x)} \sigma > \delta\,(r_x)^{\frac{q}{n-1}}\, \left(\mathrm{V}_x(r_x)\right)^{\frac{n-q-1}{n-1}}. \tag{2.19}$$

By the Hölder inequality we have

$$\int_{B_x(r_x)} \sigma \leq \left(\int_{B_x(r_x)} \sigma^{\frac{n-1}{q}} \right)^{\frac{q}{n-1}} \left(\mathrm{V}_x\left(r_x\right) \right)^{\frac{n-q-1}{n-1}}.$$

Thus, by (2.19) we get

$$\int_{B_x(r_x)} \sigma^{\frac{n-1}{q}} \geq \frac{1}{C_3} r_x \tag{2.20}$$

for any $x = \gamma(t)$ and $t \in \left[\frac{4}{3}R, \frac{5}{3}R\right]$.

By a covering argument as in [42, 43], we may choose at most countably many disjoint balls $\{B_{x_m}(r_{x_m})\}_{m\geq 1}$ with $x_m = \gamma(t_m)$, $t_m \in \left[\frac{4}{3}R, \frac{5}{3}R\right]$, each satisfying (2.20). Moreover, these balls cover at least one third of the geodesic $\gamma\left(\left[\frac{4}{3}R, \frac{5}{3}R\right]\right)$. Therefore,

$$\sum_{m\geq 1} r_{x_m} \geq \frac{1}{3}\left(\frac{5}{3}R - \frac{4}{3}R\right) = \frac{1}{9}R.$$

Together with (2.20) we have

$$\frac{1}{9}R \leq \sum_{m\geq 1} r_{x_m} \leq C_3 \sum_{m\geq 1} \int_{B_{x_m}(r_{x_m})} \sigma^{\frac{n-1}{q}} \leq C_3 \int_{B_z(R)} \sigma^{\frac{n-1}{q}},$$

where for the last inequality we have used (2.11) and that the balls $\{B_{x_m}(r_{x_m})\}_{m\geq 1}$ are disjoint in $B_z(R)$.

Combining this with (2.10) and (2.7) we conclude that

$$\frac{1}{9C_3}R \leq \int_{E(3R)\setminus E(R)} \sigma^{\frac{n-1}{q}} \leq \frac{C_2}{k}\frac{\mathrm{V}_p(3R)}{R^{n-1}}.$$

In other words,

$$\mathrm{V}_p(3R) \geq \frac{k}{C_4} R^n,$$

which contradicts (2.4) if $k > 2\,\mathrm{V}_\infty\, C_4\, 3^n$. This proves the theorem. $\square$

For a shrinking gradient Ricci soliton, the asymptotic volume ratio V_∞ is always finite. By Li-Wang [26], the following Sobolev inequality holds for dimension $n \geq 3$.

$$\left(\int_M \phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C(n)\, e^{-\frac{2\mu(g)}{n}} \int_M \left(\left|\nabla\phi\right|^2 + S\phi^2\right)$$

provided $\phi \in C_0^\infty(M)$. This implies Theorem 1.1.

Corollary 2.2. *Let (M, g) be a gradient shrinking Ricci soliton with $\alpha < \infty$, where*

$$\alpha = \limsup_{R\to\infty} \frac{1}{\mathrm{V}_p(R)} \int_{B_p(R)} \left(S\, r^2\right)^{\frac{n-1}{2}}.$$

Then the number of ends of M is bounded from above by $\Gamma(n, \alpha, \mu(g))$, a constant depending only on dimension n, $\mu(g)$ and α.

For a submanifold M in Euclidean space $\mathbb{R}^N$, the well known Michael-Simon inequality [2, 27] states that

$$\left(\int_M |\phi|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C(n) \int_M \left(|\nabla\phi| + |H|\,|\phi|\right)$$

for any $\phi \in C_0^\infty(M)$, where H is the mean curvature vector of M. By Theorem 2.1, we have the following conclusion.

Corollary 2.3. *Let M^n be a complete submanifold of $\mathbb{R}^N$ with $n \geq 2$. Suppose*

$$\tilde{\alpha} = \limsup_{R\to\infty} \frac{1}{\mathrm{V}_p(R)} \int_{B_p(R)} (r\ |H|)^{n-1} < \infty$$

and

$$\mathrm{V}_\infty = \limsup_{R\to\infty} \frac{\mathrm{V}_p(R)}{R^n} < \infty.$$

Then the number of ends of M is bounded above by a constant Γ depending only on the dimension n, $\tilde{\alpha}$ and V_∞.

Recall that a hypersurface $M \subset \mathbb{R}^{n+1}$ is a self shrinker of the mean curvature flow if it satisfies the equation

$$H = \frac{1}{2} \langle x, \mathbf{n} \rangle,$$

where x is the position vector, H the mean curvature and $\mathbf{n}$ the unit normal vector. Self shrinkers arise naturally in the singularity analysis of mean curvature flow. In fact, it follows from the monotonicity formula of Huisken [19] that tangent flows at singularities of the mean curvature flow are self shrinkers. Many examples have been constructed by gluing methods by Kapouleas, Kleene, and Möller in [20] and Nguyen in [34].

A self shrinker M is asymptotically conical if there exists a regular cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ with vertex at the origin such that the rescaled submanifold λM converges to $\mathcal{C}$ locally smoothly as $\lambda \to 0$. By a theorem of Wang [44], the limiting cone $\mathcal{C}$ uniquely determines the shrinker M.

For an asymptotically conical shrinker, clearly both $\tilde{\alpha}$ and V_∞ are finite.

Corollary 2.4. *Assume that $M^n \subset \mathbb{R}^{n+1}$ is an asymptotically conical self shrinker of the mean curvature flow of dimension $n \geq 2$. Then the number of ends $e(M) \leq \Gamma(n, \mathrm{V}_\infty, \tilde{\alpha})$, where $\tilde{\alpha}$ is defined in Corollary 2.3.*

We would also like mention a recent result of Sun-Wang [41] which bounds $e(M)$ in terms of the entropy and genus when $n = 2$.

3. Ends and solutions to Schrödinger equations

In this section we prove Theorem 1.4. The standing assumption in this section is that M is complete and that σ is a nonnegative, but not identically zero, smooth function on M.

We first recall an interior gradient estimate for positive solution u of $\Delta u = \sigma u$ established by Cheng and Yau (see Theorem 6 in [9]).

Lemma 3.1. *Suppose $u > 0$ is a solution to $\Delta u = \sigma u$ on the geodesic ball $B_p(2r)$ centered at $p \in M$ and of radius $2r$. Then*

$$|\nabla \ln u| \leq C(r) \text{ on } B_p(r),$$

where $C(r)$ is a constant depending on r, σ and the Ricci curvature lower bound of M on $B_p(2r)$.

In particular, the lemma implies that on any compact subset K of $B_p(r)$, the Harnack inequality $u(x) \leq C(K)\, u(y)$ holds for $x, y \in K$ with a constant $C(K)$ independent of u.

We now construct nontrivial solutions of the equation $\Delta u = \sigma u$ when M has more than one end. In contrast to [23], there is no need to distinguish the two cases of M being parabolic or nonparabolic.

Theorem 3.2. *Let (M, g) be a complete manifold and $E_1, E_2, \cdots, E_l$ the ends of M with respect to the geodesic ball $B_p(r_0)$ with $l \geq 2$. Then for each end E_i, there exists a positive solution u_i to the equation $\Delta u_i = \sigma u_i$ on M satisfying $0 < u_i \leq 1$ on $M \backslash E_i$ and*

$$\sup_M u_i = \limsup_{x \to E_i(\infty)} u_i(x) > 1.$$

Moreover, the functions $u_1, \cdots, u_l$ are linearly independent.

Proof. We first construct the functions u_i. To ease notation, let $E = E_i$ and $F = F_i = M \backslash E_i$. As $l \geq 2$, F must be unbounded. For $R \geq r_0$, denote $E(R) = E \cap B_p(R)$ and $F(R) = F \cap B_p(R)$. Let $v_R : B_p(R) \to \mathbb{R}$ be the solution of the Dirichlet problem

$$\begin{aligned} \Delta v_R &= \sigma v_R \quad \text{in } B_p(R) \\ v_R &= 0 \quad \text{on } \partial F(R) \\ v_R &= 1 \text{ on } \partial E(R). \end{aligned}$$

Since $\sigma \geq 0$ on M, by the strong maximum principle, it follows that $0 < v_R < 1$ in $B_p(R)$. We now normalize v_R by setting

$$u_R = C_R\, v_R,$$

where

$$C_R = \left(\max_{B_p(r_0)} v_R \right)^{-1} > 1.$$

Then u_R is a solution of

$$\begin{aligned} \Delta u_R &= \sigma u_R \quad \text{in } B_p(R) \\ u_R &= 0 \quad \text{on } \partial F(R) \\ u_R &= C_R \text{ on } \partial E(R). \end{aligned}$$

In addition,

$$\max_{B_p(r_0)} u_R = 1. \tag{3.1}$$

Hence, by Lemma 3.1 and the remark following it, we conclude from (3.1) that for any fixed $0 < r < \frac{R}{2}$,

$$\sup_{B_p(r)} u_R \leq C(r)$$

and

$$\sup_{B_p(r)} |\nabla u_R| \leq C(r),$$

where $C(r)$ is a constant independent of R. It is now easy to see that a subsequence of u_R converges to a solution $u > 0$ of $\Delta u = \sigma u$ on M. Note that u can not be a constant function as σ is not identically 0.

Since $u_R = 0$ on $\partial F(R)$, the strong maximum principle implies that $\sup_{\partial E(r)} u_R$ is strictly increasing in r and $\sup_{\partial F(r)} u_R$ decreasing in r. Therefore, the same holds true for the function u. In particular, by the fact that

$$\max_{B_p(r_0)} u = 1, \tag{3.2}$$

one concludes that $0 < u \leq 1$ on $F = M \backslash E$ and

$$\sup_M u = \limsup_{x \to E(\infty)} u(x) > 1.$$

This finishes our construction of the function u_i.

We now turn to prove that the functions $u_1, \cdots, u_l$ are linearly independent. Assume that

$$\sum_{j=1}^{l} a_j u_j = 0 \tag{3.3}$$

for some constants $a_j \in \mathbb{R}$. For an arbitrary but fixed j, if u_j is unbounded on E_j, then clearly $a_j = 0$ as u_i is bounded on E_j for all $i \neq j$.

So we may assume from here on that each u_j is bounded on E_j. Let

$$S_j = \sup_{E_j} u_j > 1.$$

Then there exists a sequence $x_{j,k} \in E_j$ such that

$$\lim_{k \to \infty} (S_j - u_j)(x_{j,k}) = 0. \tag{3.4}$$

Note that $S_j - u_j > 0$ on M. In particular, there exists a constant $C_j > 0$ satisfying $S_j - u_j > \frac{1}{C_j}$ on $B_p(r_0)$. We now claim that for $i \neq j$,

$$u_i \leq C_j (S_j - u_j) \tag{3.5}$$

on E_j.

Indeed, recall from the construction that u_i is the limit of a subsequence of $u_{i,R}$ satisfying

$$\begin{aligned} \Delta u_{i,R} &= \sigma u_{i,R} \ \text{ in } B_p(R) \\ u_{i,R} &= 0 \ \text{ on } \partial F_i(R) \\ u_{i,R} &= C_{i,R} \text{ on } \partial E_i(R), \end{aligned}$$

where $F_i = M \backslash E_i$, together with

$$\max_{B_p(r_0)} u_{i,R} = 1.$$

Now the function

$$w_{i,R} = u_{i,R} - C_j (S_j - u_j)$$

satisfies $\Delta w_{i,R} \geq 0$ on $F_i(R) \setminus F_i(r_0)$ as $\sigma \geq 0$. Also, $w_{i,R} < 0$ on $\partial F_i(R) \cup \partial F_i(r_0)$. By the maximum principle, $w_{i,R} < 0$ on $F_i(R) \backslash F_i(r_0)$. After taking limit, one concludes that $u_i \leq C_j (S_j - u_j)$ on $F_i \backslash F_i(r_0)$. Since $i \neq j$ and $E_j \subset F_i \backslash F_i(r_0)$, the claim follows.

By (3.4) and (3.5) it follows that

$$\lim_{k \to \infty} u_i(x_{j,k}) = \begin{cases} 0 & \text{if } i \neq j \\ S_j & \text{if } i = j. \end{cases}$$

Plugging this into (3.3), one infers that $a_j = 0$. But j is arbitrary. This proves that $u_1, \cdots, u_l$ are linearly independent. $\square$

4. Growth estimates

Our focus in this section is on growth rate estimates for positive solutions to $\Delta u = \sigma u$. We fix a large enough positive constant R_0 and assume that the manifold M admits a proper function ρ satisfying

$$\frac{1}{2} \leq |\nabla \rho| \leq 1 \quad \text{and} \quad \Delta \rho \leq \frac{m}{\rho} \tag{4.1}$$

in the weak sense for $\rho \geq R_0$, where m is a positive constant. Denote the sublevel and level set of ρ by

$$D(r) = \{x \in M : \rho(x) < r\} \text{ and } \Sigma(r) = \{x \in M : \rho(x) = r\}$$

respectively. They are compact as ρ is proper. Denote with $\mathrm{V}(r)$ the volume of $D(r)$ and with $\mathrm{A}(r)$ the area of $\Sigma(r)$.

Definition 4.1. *A manifold (M, g) has the mean value property $(\mathcal{M})$ if there exist constants $A_0 > 0$ and $\nu > 1$ such that for any $0 < \theta \leq 1$ and $R \geq 4R_0$,*

$$\sup_{\Sigma(R)} u \leq \frac{A_0}{\theta^{2\nu}} \frac{1}{\mathrm{V}((1+\theta)R)} \int_{D((1+\theta)R)\setminus D(R_0)} u \tag{4.2}$$

holds true for any function $u > 0$ satisfying $\Delta u \geq \sigma u$ on $D(2R)\setminus D(R_0)$.

We begin with a simple observation. Integrating by parts, one immediately sees that for any C^1 function w and $r \geq R_0$,

$$\int_{D(r)} w \Delta \rho + \int_{D(r)} \langle \nabla w, \nabla \rho \rangle = \int_{\Sigma(r)} w \frac{\partial \rho}{\partial \eta}$$

where η is the unit normal vector to $\Sigma(r)$ given by $\eta = \frac{\nabla \rho}{|\nabla \rho|}$. Taking a derivative in r of this identity yields the following formula:

$$\frac{d}{dr} \int_{\Sigma(r)} w \, |\nabla \rho| = \int_{\Sigma(r)} \frac{\langle \nabla w, \nabla \rho \rangle}{|\nabla \rho|} + \int_{\Sigma(r)} \frac{w \Delta \rho}{|\nabla \rho|} \; . \tag{4.3}$$

The following lemma provides volume and area estimates.

Lemma 4.2. *Let $A(r)$ be the area of $\Sigma(r)$ and $V(r)$ the volume of $D(r)$. Then*

$$\mathrm{A}(r) \leq \frac{c(m)}{r} \mathrm{V}(r),$$
$$\mathrm{V}((1+\theta)r) \leq (1+\theta)^{c(m)} \mathrm{V}(r),$$
$$\mathrm{V}(r) \leq r^{\gamma(m)} \mathrm{V}(R_0)$$

for all $r \geq R_0$ and $0 < \theta \leq 1$, where $c(m)$ and $\gamma(m)$ depend only on m.

Proof. By the co-area formula, there exists $\frac{r}{2} < t < r$ such that

$$\begin{aligned} \mathrm{V}(r) &\geq \mathrm{Vol}\left(D(r) \setminus D\left(\frac{r}{2}\right)\right) \\ &= \frac{r}{2} \int_{\Sigma(t)} \frac{1}{|\nabla \rho|}. \end{aligned} \tag{4.4}$$

From (4.1) we have

$$\Delta \rho \leq \frac{4m}{\rho} \left| \nabla \rho \right|^2$$

for all $r \geq R_0$. Hence, applying (4.3) with $w = 1$ implies

$$\begin{aligned} \frac{d}{dr} \int_{\Sigma(r)} \left| \nabla \rho \right| &= \int_{\Sigma(r)} \frac{\Delta \rho}{\left| \nabla \rho \right|} \\ &\leq \frac{4m}{r} \int_{\Sigma(r)} \left| \nabla \rho \right|. \end{aligned}$$

Integrating in r we conclude that

$$\begin{aligned} \int_{\Sigma(r)} \left| \nabla \rho \right| &\leq \left(\frac{r}{t} \right)^{4m} \int_{\Sigma(t)} \left| \nabla \rho \right| \\ &\leq \left(\frac{r}{t} \right)^{4m} \int_{\Sigma(t)} \frac{1}{\left| \nabla \rho \right|}. \end{aligned}$$

Together with (4.4), this implies

$$\int_{\Sigma(r)} \left| \nabla \rho \right| \leq \frac{c(m)}{r} \mathrm{V}\left(r \right). \tag{4.5}$$

Now the area estimate follows from (4.1).

Note that (4.5) and (4.1) also imply

$$\mathrm{V}'\left(r \right) \leq \frac{c(m)}{r} \mathrm{V}(r).$$

Integrating in r we obtain

$$\mathrm{V}(R) \leq \left(\frac{R}{r} \right)^{c(m)} \mathrm{V}(r) \tag{4.6}$$

for all $R_0 < r < R$. Clearly, it gives both the volume doubling property and growth estimate. This proves the result. □

The next lemma is our starting point for establishing growth estimates for positive solutions to $\Delta u = \sigma u$.

Lemma 4.3. *A positive solution u of $\Delta u = \sigma u$ on $D(R) \backslash D(R_0)$ satisfies*

$$\frac{d}{dr} \left(\frac{1}{r^{4m}} \int_{\Sigma(r)} u \left| \nabla \rho \right| \right) \leq \frac{1}{r^{4m}} \int_{D(r) \backslash D(r_0)} \sigma u + \frac{1}{r^{4m}} \int_{\Sigma(r_0)} \frac{\langle \nabla u, \nabla \rho \rangle}{\left| \nabla \rho \right|}$$

for all $R \geq r \geq r_0 \geq R_0$.

Proof. Applying (4.3) to $w = u$ and taking into account that

$$\int_{\Sigma(r)} \frac{\langle \nabla u, \nabla \rho \rangle}{\left| \nabla \rho \right|} = \int_{D(r) \backslash D(r_0)} \Delta u + \int_{\Sigma(r_0)} \frac{\langle \nabla u, \nabla \rho \rangle}{\left| \nabla \rho \right|},$$

we obtain

$$\frac{d}{dr} \int_{\Sigma(r)} u \left| \nabla \rho \right| = \int_{D(r) \backslash D(r_0)} \sigma u + \int_{\Sigma(r)} u \frac{\Delta \rho}{\left| \nabla \rho \right|} + \int_{\Sigma(r_0)} \frac{\langle \nabla u, \nabla \rho \rangle}{\left| \nabla \rho \right|}. \tag{4.7}$$

By (4.1) we have that

$$\int_{\Sigma(r)} u\frac{\Delta\rho}{|\nabla\rho|} \leq \frac{m}{r}\int_{\Sigma(r)} \frac{u}{|\nabla\rho|} \leq \frac{4m}{r}\int_{\Sigma(r)} u\,|\nabla\rho|$$

for $r \geq r_0 \geq R_0$. Plugging this into (4.7) implies

$$\frac{d}{dr}\int_{\Sigma(r)} u\,|\nabla\rho| \leq \int_{D(r)\setminus D(r_0)} \sigma u + \frac{4m}{r}\int_{\Sigma(r)} u\,|\nabla\rho| + \int_{\Sigma(r_0)} \frac{\langle\nabla u, \nabla\rho\rangle}{|\nabla\rho|}.$$

This proves the result. $\square$

We now prove a preliminary growth estimate by imposing a pointwise quadratic decay assumption on σ of the form

$$\sigma \leq \frac{\Upsilon}{\rho^2} \quad \text{on } M\setminus D(r_0), \tag{4.8}$$

where $r_0 \geq 4R_0$ and $\Upsilon > 0$ is a constant.

Proposition 4.4. *Assume that (M, g) admits a proper function ρ satisfying (4.1) and has the mean value property $(\mathcal{M})$. If σ decays quadratically as in (4.8), then there exists a constant $C = C(m, \Upsilon) > 0$ such that*

$$u \leq (\rho+1)^C \sup_{D(r_0)\setminus D(R_0)} u \quad \text{on } D\left(\frac{R}{2}\right)\setminus D(R_0)$$

for any positive solution of $\Delta u = \sigma u$ on $D(R)\setminus D(R_0)$ with $R \geq r_0$.

Proof. The result is obvious if $R \leq 2r_0$. Hence, we may assume from now on that $R > 2r_0$. By Lemma 3.1, it follows that there exists $C(r_0) > 0$ such that

$$\left|\int_{\Sigma(r_0)} \frac{\langle\nabla u, \nabla\rho\rangle}{|\nabla\rho|}\right| \leq C(r_0) \sup_{\Sigma(r_0)} u \tag{4.9}$$

with the constant $C(r_0)$ independent of u.

By normalizing u if necessary, we may assume that

$$\sup_{D(r_0)\setminus D(R_0)} u = 1. \tag{4.10}$$

So we get

$$\left|\int_{\Sigma(r_0)} \frac{\langle\nabla u, \nabla\rho\rangle}{|\nabla\rho|}\right| \leq C(r_0). \tag{4.11}$$

By Lemma 4.3 and (4.1) we have that

$$\begin{aligned} \frac{d}{dr}\left(\frac{1}{r^{4m}}\int_{\Sigma(r)} u\,|\nabla\rho|\right) &\leq \frac{1}{r^{4m}}\int_{D(r)\setminus D(r_0)} \sigma u + \frac{1}{r^{4m}}\int_{\Sigma(r_0)} \frac{\langle\nabla u, \nabla\rho\rangle}{|\nabla\rho|} \\ &\leq \frac{4}{r^{4m}}\int_{D(r)\setminus D(r_0)} \sigma u\,|\nabla\rho|^2 + \frac{1}{r^{4m}}\int_{\Sigma(r_0)} \frac{\langle\nabla u, \nabla\rho\rangle}{|\nabla\rho|} \end{aligned} \tag{4.12}$$

for all $r \in [r_0, R]$.

Combining (4.12), (4.11) and (4.8), we conclude

$$\frac{d}{dr}\left(\frac{1}{r^{4m}}\int_{\Sigma(r)} u\,|\nabla\rho|\right) \leq \frac{4\Upsilon}{r^{4m}}\int_{D(r)\setminus D(r_0)} u\frac{|\nabla\rho|^2}{\rho^2} + \frac{C(r_0)}{r^{4m}} \tag{4.13}$$

for all $r \in [r_0, R]$. If we set

$$\omega(r) = \int_{D(r)\setminus D(r_0)} u \frac{|\nabla \rho|^2}{\rho^2}, \tag{4.14}$$

then the co-area formula gives

$$\omega'(r) = \frac{1}{r^2} \int_{\Sigma(r)} u |\nabla \rho|.$$

So (4.13) becomes

$$\frac{d}{dr}\left(\frac{1}{r^{4m-2}}\omega'(r)\right) \leq \frac{4\Upsilon}{r^{4m}}\omega(r) + \frac{C(r_0)}{r^{4m}}$$

or

$$r^2\omega''(r) - (4m-2)\, r\omega'(r) - 4\Upsilon\omega(r) \leq C(r_0) \tag{4.15}$$

for all $r \in [r_0, R]$. Direct calculation then implies that the function

$$\xi(r) = r^a \omega(r) \tag{4.16}$$

satisfies

$$r\xi''(r) - (2a + 4m - 2)\,\xi'(r) \leq C(r_0)\, r^{a-1} \tag{4.17}$$

for all $r \in [r_0, R]$, where

$$a = \frac{\sqrt{(4m-1)^2 + 16\Upsilon} - (4m-1)}{2}. \tag{4.18}$$

Rewriting (4.17) into

$$\frac{d}{dr}\left(\frac{\xi'(r)}{r^{2a+4m-2}}\right) \leq \frac{C(r_0)}{r^{a+4m}}$$

and integrating from r_0 to r, we get

$$\xi'(r) \leq \left(\frac{r}{r_0}\right)^{2a+4m-2} \xi'(r_0) + C(r_0) r^{2a+4m-2} \tag{4.19}$$

for all $r \in [r_0, R]$.

According to (4.16) and (4.14) we have

$$\xi'(r_0) = r_0^{a-2} \int_{\Sigma(r_0)} u |\nabla \rho|.$$

Hence, by (4.10),

$$\xi'(r_0) \leq C(r_0).$$

Plugging into (4.19) we conclude that

$$\xi'(r) \leq C(r_0) r^{2a+4m-2}$$

for all $r \in [r_0, R]$. After integrating from r_0 to r, this immediately leads to

$$\omega(r) \leq C(r_0) r^{a+4m-1}.$$

In view of (4.14) and (4.18), we have

$$\int_{D(r)\setminus D(R_0)} u \leq C(r_0) r^{C(m,\Upsilon)}$$

for all $r\in[r_0,R]$. Finally, the mean value property implies that

$$\sup_{\Sigma(\frac{1}{2}r)} u\leq C(A,\mu,r_0)r^{C(m,\Upsilon)}$$

for all $r\in[2r_0,R]$. This proves the result. $\square$

We remark that the assumption of σ being of quadratic decay is optimal in the sense that any slower decay will render the result to fail. Indeed, on Euclidean space, the function $u(x)=\exp\left(r^{\epsilon}(x)\right)$ satisfies the equation $\Delta u=\sigma u$ with σ decaying of order $2-2\epsilon$.

Our main result of this section is that the order of polynomial growth of u in fact only depends on an integral quantity of the function σ provided that u is a priori of polynomial growth, namely,

$$|u|\leq\rho^{C}\quad\text{on } M\backslash D(R_0)$$

for some constant $C>0$.

In the following, we denote

$$\alpha=\limsup_{R\to\infty}\left(R^{2q}⨍_{\Sigma(R)}\sigma^{q}\right)^{\frac{1}{q}}$$

with $q\geq 1$ to be specified.

Theorem 4.5. *Assume that (M,g) admits a proper function ρ satisfying (4.1) and has the mean value property $(\mathcal{M})$. For a positive function u of polynomial growth, satisfying $\Delta u=\sigma u$ on $M\backslash D(R_0)$, if $\alpha<\infty$ for some $q>\nu-\frac{1}{2}$, then there exists a constant $\Gamma(m,A_0,\nu,\alpha)>0$ such that*

$$u\leq\Lambda\left(\rho^{\Gamma}+1\right)\quad\text{on } M\backslash D(R_0),$$

where $\Lambda>0$ is a constant depending on u. The same estimate for u holds true in the case $q=\nu-\frac{1}{2}$ with $\Gamma=\Gamma(m,A_0,\nu)$ provided that $\alpha\leq\alpha_0(m,A_0,\nu)$, a sufficiently small positive constant.

Proof. By the Hölder inequality, α is increasing in q. So we may restrict our attention to those q that

$$0\leq\varepsilon<\frac{1}{2},$$

where

$$\varepsilon=\frac{2q+1-2\nu}{q}. \tag{4.20}$$

To treat both cases $q>\nu-\frac{1}{2}$ and $q=\nu-\frac{1}{2}$ at the same time, we let

$$\bar{\alpha}=\min\{\alpha,1\}\quad\text{and } \widetilde{\alpha}=\max\{\alpha,1\}. \tag{4.21}$$

Note that $\alpha=\bar{\alpha}\,\widetilde{\alpha}$. In the following,

$$C_0=C_0\left(m,A_0,\nu,\widetilde{\alpha}\right)>1 \tag{4.22}$$

is a fixed large constant, depending only on m,A_0,ν and $\widetilde{\alpha}$, to be specified later.

In view of the definition of α, there exists $r_0\geq 4R_0$ such that

$$\int_{\Sigma(r)}\frac{\sigma^{q}}{|\nabla\rho|}\leq 3\alpha^{q}r^{-2q}\mathrm{A}(r)$$

for all $r \geq r_0$. From Lemma 4.2 it follows that

$$\int_{\Sigma(r)} \frac{\sigma^q}{|\nabla\rho|} \leq c\,(m)\,\alpha^q r^{-2q-1}\mathrm{V}(r), \tag{4.23}$$

for all $r \geq r_0$.

Denote

$$\chi(r) = \int_{D(r)\setminus D(R_0)} u\frac{|\nabla\rho|^2}{\rho^{4m}}. \tag{4.24}$$

We claim that χ satisfies the following inequality.

$$r^{4m}\chi''(r) \leq \frac{C_0\bar{\alpha}}{\theta^{\frac{2\nu}{q}}}\int_{r_0}^{r} \chi^{\frac{1}{q}}((1+\theta)t)(\chi'(t))^{1-\frac{1}{q}} t^{4m-2-\frac{1}{q}}\,dt + \Lambda_0 \tag{4.25}$$

for all $r \geq r_0$ and $0 < \theta \leq 1$, where

$$\Lambda_0 = \int_{\Sigma(r_0)} (u + |\nabla u|)\,. \tag{4.26}$$

We first prove (4.25) for $q > 1$. By the co-area formula,

$$\chi'(r) = \frac{1}{r^{4m}}\int_{\Sigma(r)} u\,|\nabla\rho|\,. \tag{4.27}$$

Hence, using Lemma 4.3, we have

$$\begin{aligned}\chi''(r) &= \frac{d}{dr}\left(\frac{1}{r^{4m}}\int_{\Sigma(r)} u\,|\nabla\rho|\right)\\ &\leq \frac{1}{r^{4m}}\int_{D(r)\setminus D(r_0)} \sigma u + \frac{1}{r^{4m}}\int_{\Sigma(r_0)} \frac{\langle\nabla u, \nabla\rho\rangle}{|\nabla\rho|}.\end{aligned} \tag{4.28}$$

The first term can be estimated by the co-area formula and Hölder inequality as

$$\begin{aligned}\int_{D(r)\setminus D(r_0)} \sigma u &= \int_{r_0}^{r}\left(\int_{\Sigma(t)} \frac{\sigma u}{|\nabla\rho|}\right)dt\\ &\leq \int_{r_0}^{r}\left(\int_{\Sigma(t)} \frac{\sigma^q}{|\nabla\rho|}\right)^{\frac{1}{q}}\left(\int_{\Sigma(t)} \frac{u^p}{|\nabla\rho|}\right)^{\frac{1}{p}}dt,\end{aligned} \tag{4.29}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Invoking (4.23) we conclude

$$\int_{D(r)\setminus D(r_0)} \sigma u \leq c(m)\alpha\int_{r_0}^{r}\left(\int_{\Sigma(t)} \frac{u^p}{|\nabla\rho|}\right)^{\frac{1}{p}} \frac{\mathrm{V}(t)^{\frac{1}{q}}}{t^{2+\frac{1}{q}}}dt. \tag{4.30}$$

On the other hand, the mean value property (4.2) implies

$$\begin{aligned}\sup_{\Sigma(t)} u &\leq \frac{A_0}{\theta^{2\nu}}\frac{1}{\mathrm{V}((1+\theta)t)}\int_{D((1+\theta)t)\setminus D(R_0)} u\\ &\leq \frac{4A_0}{\theta^{2\nu}}\frac{((1+\theta)\,t)^{4m}}{\mathrm{V}(t)}\int_{D((1+\theta)t)\setminus D(R_0)} u\frac{|\nabla\rho|^2}{\rho^{4m}}\\ &\leq \frac{c(m)A_0}{\theta^{2\nu}}\frac{t^{4m}}{\mathrm{V}(t)}\chi((1+\theta)t)\end{aligned} \tag{4.31}$$

for all $t \geq r_0$. Therefore,

$$\begin{aligned}
\left(\int_{\Sigma(t)} \frac{u^p}{|\nabla\rho|}\right)^{\frac{1}{p}} &\leq \left(\sup_{\Sigma(t)} u\right)^{\frac{1}{q}} \left(\int_{\Sigma(t)} \frac{u}{|\nabla\rho|}\right)^{\frac{1}{p}} \\
&\leq \frac{c(m)A_0^{\frac{1}{q}}}{\theta^{\frac{2\nu}{q}}} \frac{t^{\frac{4m}{q}}}{\mathrm{V}(t)^{\frac{1}{q}}} \chi^{\frac{1}{q}}((1+\theta)t) \left(\int_{\Sigma(t)} \frac{u}{|\nabla\rho|}\right)^{\frac{1}{p}} \\
&\leq \frac{c(m)A_0^{\frac{1}{q}}}{\theta^{\frac{2\nu}{q}}} \frac{t^{4m}}{\mathrm{V}(t)^{\frac{1}{q}}} \chi^{\frac{1}{q}}((1+\theta)t) \left(\chi'(t)\right)^{\frac{1}{p}},
\end{aligned}$$

where in the last line we have used (4.27).

Plugging this into (4.30) we conclude that

$$\int_{D(r)\setminus D(r_0)} \sigma u \leq \frac{C_0\bar{\alpha}}{\theta^{\frac{2\nu}{q}}} \int_{r_0}^{r} \chi^{\frac{1}{q}}((1+\theta)t) \left(\chi'(t)\right)^{\frac{1}{p}} t^{4m-2-\frac{1}{q}} dt, \tag{4.32}$$

where $C_0 = c(m)A_0^{\frac{1}{q}}\widetilde{\alpha}$ for some $c(m)$ depending only on m.

By (4.28) and (4.32) it follows that

$$\begin{aligned}
\chi''(r) \leq\ & \frac{C_0\bar{\alpha}}{\theta^{\frac{2\nu}{q}} r^{4m}} \int_{r_0}^{r} \chi^{\frac{1}{q}}((1+\theta)t) \left(\chi'(t)\right)^{\frac{1}{p}} t^{4m-2-\frac{1}{q}} dt \\
&+ \frac{1}{r^{4m}} \int_{\Sigma(r_0)} \frac{\langle \nabla u, \nabla\rho\rangle}{|\nabla\rho|}.
\end{aligned}$$

In view of (4.26), this can be rewritten into

$$r^{4m}\chi''(r) \leq \frac{C_0\bar{\alpha}}{\theta^{\frac{2\nu}{q}}} \int_{r_0}^{r} \chi^{\frac{1}{q}}((1+\theta)t) \left(\chi'(t)\right)^{1-\frac{1}{q}} t^{4m-2-\frac{1}{q}} dt + \Lambda_0.$$

Hence, (4.25) holds for any $q > 1$.

To extend the result to $q = 1$, we simply let $q \to 1$ in (4.25) and note that both sides are continuous as functions of q.

In conclusion, we have

$$r^{4m}\chi''(r) \leq \frac{C_0\bar{\alpha}}{\theta^{\frac{2\nu}{q}}} \int_{r_0}^{r} \chi^{\frac{1}{q}}((1+\theta)t) \left(\chi'(t)\right)^{1-\frac{1}{q}} t^{4m-2-\frac{1}{q}} dt + \Lambda_0 \tag{4.33}$$

for all $r \geq r_0$ and $0 < \theta \leq 1$.

Since u is assumed to be of polynomial growth, there exist constants $\bar{b} > 0$ and $\bar{\Lambda} > 0$ such that

$$u \leq \bar{\Lambda}\rho^{\bar{b}} \quad \text{on } M\setminus D(r_0).$$

Together with Lemma 4.2 we get

$$\chi'(r) = \frac{1}{r^{4m}} \int_{\Sigma(r)} u\,|\nabla\rho| \leq c(m)\bar{\Lambda}\, r^{\bar{b}+\gamma(m)}\mathrm{V}(R_0).$$

Therefore, for $r \geq r_0$,

$$\chi'(r) \leq \Lambda r^{b} \tag{4.34}$$

for some constants $b > 0$ and $\Lambda > 0$.

Obviously, the constant b in (4.34) can be chosen in such a way that (4.34) no longer holds with b replaced by $b-1$ for whatever constant Λ. Also, the constant Λ can be arranged to satisfy that $\Lambda \geq \Lambda_0$ and

$$\Lambda \geq \int_{D(r_0)\setminus D(R_0)} (u+|\nabla u|). \tag{4.35}$$

For ε in (4.20) and $C_0 = C_0\left(m, A_0, \nu, \widetilde{\alpha}\right)$ from (4.33) we assume by contradiction that

$$\min\left\{\frac{b^{\varepsilon}}{\bar{\alpha}}, b\right\} > \left(100 C_0\right)^2. \tag{4.36}$$

We now prove by induction on $k \geq 0$ that

$$\chi'(r) \leq \Lambda\bigg(\left(\frac{\bar{\alpha}}{b^{\varepsilon}}\right)^{\frac{k}{2}} r^b + r^{b-1}\bigg) \tag{4.37}$$

for all $r \geq r_0$.

Clearly, (4.37) holds for $k=0$ in view of (4.34). We assume it is true for k and prove it for $k+1$. Integrating (4.37) we obtain that

$$\begin{aligned}
\chi(r) &\leq \Lambda \int_{r_0}^{r} \bigg(\left(\frac{\bar{\alpha}}{b^{\varepsilon}}\right)^{\frac{k}{2}} t^b + t^{b-1}\bigg) dt + \chi(r_0) \\
&\leq \frac{\Lambda}{b}\bigg(\left(\frac{\bar{\alpha}}{b^{\varepsilon}}\right)^{\frac{k}{2}} r^{b+1} + r^b\bigg) + \Lambda,
\end{aligned}$$

where the last line follows from (4.35). Since

$$\Lambda \leq \frac{\Lambda}{b} r^b,$$

this implies

$$\chi(r) \leq \frac{2\Lambda}{b}\bigg(\left(\frac{\bar{\alpha}}{b^{\varepsilon}}\right)^{\frac{k}{2}} r^{b+1} + r^b\bigg)$$

for all $r \geq r_0$. Therefore,

$$\chi((1+\theta)r) \leq \frac{2\Lambda}{b}(1+\theta)^{b+1}\bigg(\left(\frac{\bar{\alpha}}{b^{\varepsilon}}\right)^{\frac{k}{2}} r^{b+1} + r^b\bigg) \tag{4.38}$$

for all $r \geq r_0$ and $0 < \theta \leq 1$.

By (4.37) and (4.38) we get

$$\begin{aligned}
&\int_{r_0}^{r} \chi^{\frac{1}{q}}((1+\theta)t)\left(\chi'(t)\right)^{1-\frac{1}{q}} t^{4m-2-\frac{1}{q}} dt \\
&\qquad \leq \frac{2\Lambda}{b^{\frac{1}{q}}}(1+\theta)^{\frac{b+1}{q}} \int_{r_0}^{r} \bigg(\left(\frac{\bar{\alpha}}{b^{\varepsilon}}\right)^{\frac{k}{2}} t^b + t^{b-1}\bigg) t^{4m-2} dt \\
&\qquad \leq \frac{2\Lambda}{b^{1+\frac{1}{q}}}(1+\theta)^{\frac{b+1}{q}} \bigg(\left(\frac{\bar{\alpha}}{b^{\varepsilon}}\right)^{\frac{k}{2}} r^{b+4m-1} + r^{b+4m-2}\bigg).
\end{aligned}$$

Plugging into (4.33), we arrive at

$$\chi''(r) \leq \frac{2\Lambda C_0 \bar{\alpha}}{\theta^{\frac{2\nu}{q}} b^{1+\frac{1}{q}}}(1+\theta)^{\frac{b+1}{q}} \bigg(\left(\frac{\bar{\alpha}}{b^{\varepsilon}}\right)^{\frac{k}{2}} r^{b-1} + r^{b-2}\bigg) + \frac{\Lambda_0}{r^{4m}}$$

for all $r \geq r_0$. Integrating in r then yields

$$\chi'(r) \leq \frac{3\Lambda C_0 \bar{\alpha}}{\theta^{\frac{2\nu}{q}} b^{2+\frac{1}{q}}}(1+\theta)^{\frac{b+1}{q}} \bigg(\left(\frac{\bar{\alpha}}{b^{\varepsilon}}\right)^{\frac{k}{2}} r^{b} + r^{b-1}\bigg) + \frac{1}{2}\Lambda_0 + \chi'\left(r_0\right) \tag{4.39}$$

for all $r \geq r_0$ and $0 < \theta \leq 1$. Note that by (4.26)

$$\chi'(r_0) = \frac{1}{r_0^{4m}} \int_{\Sigma(r_0)} u\,|\nabla \rho| \leq \frac{1}{2}\Lambda_0.$$

Setting $\theta = \frac{1}{b}$ in (4.39) and using (4.20), we obtain that

$$\chi'(r) \leq 4eC_0 \frac{\bar{\alpha}}{b^{\varepsilon}} \Lambda \left(\left(\frac{\bar{\alpha}}{b^{\varepsilon}} \right)^{\frac{k}{2}} r^b + r^{b-1} \right) + \Lambda_0.$$

In view of (4.36),

$$4eC_0 \frac{\bar{\alpha}}{b^{\varepsilon}} \leq \frac{1}{2} \left(\frac{\bar{\alpha}}{b^{\varepsilon}} \right)^{\frac{1}{2}}.$$

Hence, the preceding inequality becomes

$$\chi'(r) \leq \frac{1}{2}\Lambda \left(\left(\frac{\bar{\alpha}}{b^{\varepsilon}} \right)^{\frac{k+1}{2}} r^b + r^{b-1} \right) + \Lambda_0.$$

However,

$$\Lambda_0 \leq \Lambda \leq \frac{1}{2}\Lambda r^{b-1}$$

for $r \geq r_0$. In conclusion,

$$\chi'(r) \leq \Lambda \left(\left(\frac{\bar{\alpha}}{b^{\varepsilon}} \right)^{\frac{k+1}{2}} r^b + r^{b-1} \right)$$

for all $r \geq r_0$.

This completes the induction step and proves that (4.37) holds for all $k \geq 0$. We have thus established that

$$\chi'(r) \leq \Lambda \left(\left(\frac{\bar{\alpha}}{b^{\varepsilon}} \right)^{\frac{k}{2}} r^b + r^{b-1} \right) \tag{4.40}$$

for all $k \geq 0$ and all $r \geq r_0$.

By (4.36) we have $\frac{\bar{\alpha}}{b^{\varepsilon}} < 1$. Hence, by letting $k \to \infty$ in (4.40) one sees that

$$\chi'(r) \leq \Lambda r^{b-1}$$

for all $r \geq r_0$. This clearly contradicts with the choice of b.

In conclusion, we must have

$$\min \left\{ \frac{b^{\varepsilon}}{\bar{\alpha}}, b \right\} \leq (100C_0)^2 \tag{4.41}$$

for some constant $C_0 = C_0(m, A_0, \nu, \widetilde{\alpha})$.

Let us consider first the case $q > \nu - \frac{1}{2}$ or $\varepsilon > 0$. It is easy to see from (4.41) that

$$b \leq (100C_0)^{\frac{2}{\varepsilon}}.$$

Therefore,

$$\int_{\Sigma(r)} \frac{u}{|\nabla \rho|} \leq \Lambda r^{\Gamma_{\epsilon}-1}$$

for all $r \geq r_0$, where

$$\Gamma_{\varepsilon} = (100C_0)^{\frac{2}{\varepsilon}} + 4m + 1.$$

Integrating in r and applying the mean value inequality (4.2), we get

$$u \leq \widetilde{\Lambda} \rho^{\Gamma_{\epsilon}} \quad \text{on } M \backslash D(r_0), \tag{4.42}$$

where $\widetilde{\Lambda} = \frac{2^{\Gamma_{\varepsilon}} \Lambda}{\mathrm{V}(R_0)}$.

Assume now that $q = \nu - \frac{1}{2}$ or $\varepsilon = 0$. Then (4.41) implies

$$\min\left\{\frac{1}{\bar{\alpha}}, b\right\} \leq (100C_0)^2 . \tag{4.43}$$

So if $\alpha < \alpha_0$ with

$$\frac{1}{\alpha_0} = (100C_0)^2 ,$$

then

$$\frac{1}{\bar{\alpha}} = \frac{1}{\alpha} > (100C_0)^2$$

and (4.43) implies that

$$b \leq (100C_0)^2 .$$

As above, we conclude that

$$u \leq \widetilde{\Lambda}\rho^{\Gamma} \quad \text{on } M \backslash D(r_0) \tag{4.44}$$

for some $\Gamma(m, A_0, \nu)$, where $\widetilde{\Lambda} = \frac{2^{\Gamma}\Lambda}{\mathrm{V}(R_0)}$.

By (4.42) and (4.44), the theorem is proved. □

Combining Proposition 4.4 with Theorem 4.5, we have the following corollary concerning positive solutions u to $\Delta u = \sigma u$ on $M \setminus D(R_0)$.

Corollary 4.6. *Assume that (M, g) admits a proper function ρ satisfying (4.1) and has the mean value property $(\mathcal{M})$. Suppose that σ decays quadratically. Then there exists $\Gamma(m, A_0, \nu, \alpha) > 0$ such that*

$$u \leq \Lambda\left(\rho^{\Gamma} + 1\right) \quad \text{on } M \backslash D(R_0),$$

where $\Lambda > 0$ is a constant depending on u, provided that $\alpha < \infty$ for some $q > \nu - \frac{1}{2}$. In the case $q = \nu - \frac{1}{2}$, the same conclusion holds for some $\Gamma(m, A_0, \nu) > 0$ when $\alpha \leq \alpha_0(m, A_0, \nu)$, a sufficiently small positive constant.

5. Dimension Estimate

In this section, we establish a dimension estimate for the space $\mathcal{P}$ spanned by all positive solutions to the equation $\Delta u = \sigma u$ on M. We continue to assume that M admits a proper function ρ satisfying (4.1) and has the mean value property $(\mathcal{M})$. Our argument closely follows that in [21].

Define

$$L^d(M) = \left\{v : \Delta v = \sigma v, |v| \leq c\,\rho^d \ \text{ on } M\right\},$$

the space of polynomial growth solutions of degree at most d.

Lemma 5.1. *Assume that (M, g) admits a proper function ρ satisfying (4.1) and has the mean value property $(\mathcal{M})$. Then $\dim L^d(M) \leq \Gamma(m, A_0, \nu, d)$.*

Proof. Let $\mathcal{W}_l$ be any l-dimensional subspace of $L^d(M)$, where $l > 1$. For $R > 0$, define the inner product

$$A_R(u, v) = \int_{D(R)} u\, v$$

for $u, v \in \mathcal{W}_l$. We claim that there exists $R > 4R_0$ large enough so that for $\{u_1, \cdots, u_l\}$, an orthonormal basis of $\mathcal{W}_l$ with respect to A_{2R},

$$\sum_{i=1}^{l} \int_{D(R)} u_i^2 \geq \frac{l}{\overline{\Gamma}}, \tag{5.1}$$

where $\bar{\Gamma} = 2^{\gamma(m)+2\,d+1}$ with $\gamma(m)$ being the same constant from Lemma 4.2.

Indeed, assume by contradiction that (5.1) fails for all $R > 4R_0$. To simplify notation, for $R_2 > R_1$, we denote by

$$\mathrm{tr}_{A_{R_2}} A_{R_1} = \sum_{i=1}^{l} \int_{D(R_1)} v_i^2$$

for orthonormal basis $\{v_1, \cdots, v_l\}$ with respect to A_{R_2}. Since (5.1) fails for all $R > 4R_0$, we have that

$$\frac{1}{\bar{\Gamma}} > \frac{\mathrm{tr}_{A_{2R}} A_R}{l} \geq (\det_{A_{2R}} A_R)^{\frac{1}{l}},$$

where the last estimate follows from the arithmetic-geometric mean inequality. In other words,

$$\det_{A_{2R}} A_R \leq \frac{1}{\bar{\Gamma}^l} \tag{5.2}$$

for all $R \geq 4R_0$. Iterating (5.2) and using that

$$(\det_{A_T} A_R)\,(\det_{A_R} A_S) = \det_{A_T} A_S,$$

we get

$$\det_{A_{2^j R}} A_R \leq \frac{1}{\bar{\Gamma}^{lj}}.$$

Equivalently,

$$\det_{A_R} A_{2^j R} \geq \bar{\Gamma}^{lj} \tag{5.3}$$

for all $j > 0$ and $R \geq 4R_0$.

On the other hand, Lemma 4.2 implies that $\mathrm{V}(2^j R) \leq (2^j R)^{\gamma(m)} \mathrm{V}(R_0)$. Together with the fact that $u \in \mathcal{W}_l$ is of polynomial growth of order at most d, we conclude

$$\det_{A_R} A_{2^j R} \leq \Lambda^{2l} (2^j R)^{(\gamma(m)+2\,d)l} \mathrm{V}(R_0)^l.$$

As $\bar{\Gamma} > 2^{\gamma(m)+2\,d}$, this contradicts (5.3) after letting $j \to \infty$. This proves (5.1).

For $x \in \Sigma(R)$ we note that there exists a subspace $\mathcal{W}_x$ of $\mathcal{W}_l$, of codimension at most one, such that $u(x) = 0$ for all $u \in \mathcal{W}_x$. So one may choose an orthonormal basis in $\mathcal{W}_l$ with $u_2, \cdots, u_l \in \mathcal{W}_x$. By the mean value property $(\mathcal{M})$ we get

$$\begin{aligned}\sum_{i=1}^{l} u_i^2(x) &= u_1^2(x) \\ &\leq \frac{C(A_0,\mu)}{\mathrm{V}(2R)} \int_{D(2R)} u_1^2 \\ &= \frac{C(A_0,\mu)}{\mathrm{V}(2R)}.\end{aligned}$$

The function $\Psi(x) = \sum_{i=1}^{l} u_i^2(x)$ is subharmonic, therefore its maximum on $D(R)$ is achieved on $\Sigma(R)$. We have thus proved that

$$\sum_{i=1}^{l} u_i^2(x) \leq \frac{C(A_0,\mu)}{\mathrm{V}(2R)}$$

for $x \in D(R)$. Together with (5.1) we get

$$\begin{aligned}\frac{l}{\bar{\Gamma}} &\leq \sum_{i=1}^{l} \int_{D(R)} u_i^2(x)\\ &\leq \frac{C(A_0,\mu)}{\mathrm{V}(2R)}\mathrm{V}(R).\end{aligned}$$

Therefore,

$$l \leq C(A_0,\mu)\bar{\Gamma}.$$

Since this holds true for any l-dimensional subspace $\mathcal{W}_l$ of $L^d(M)$, we conclude that

$$\dim L^d(M) \leq C(A_0,\mu)\bar{\Gamma}$$

as well. This proves the result. □

Summarizing, we have the following theorem. Recall $\mathcal{P}$ is the space spanned by all positive solutions to the equation $\Delta u = \sigma u$.

Theorem 5.2. *Assume that (M,g) admits a proper function ρ satisfying (4.1) and has the mean value property $(\mathcal{M})$. Suppose that σ decays quadratically. Then $\dim \mathcal{P} \leq \Gamma(m, A_0, \nu, \alpha)$ provided that $\alpha < \infty$ for some $q > \nu - \frac{1}{2}$. In the case $q = \nu - \frac{1}{2}$, the same conclusion holds for some $\Gamma(m, A_0, \nu)$ when $\alpha \leq \alpha_0(m, A_0, \nu)$, a sufficiently small positive constant. Consequently, the number of ends $e(M)$ of M satisfies the same estimate as well.*

Proof. According to Theorem 3.2, the number of ends $e(M)$ is at most the dimension of $\mathcal{P}$. However, Corollary 4.6 implies that $\mathcal{P} \subset L^d(M)$ with $d = \Gamma(m, A_0, \nu, \alpha)$ in the case $q > \nu - \frac{1}{2}$ and $d = \Gamma(m, A_0, \nu)$ in the case $q = \nu - \frac{1}{2}$, respectively. The conclusion on the dimension estimate of $\mathcal{P}$ then follows from Lemma 5.1. This proves the theorem. □

6. Sobolev inequality

In this section, we show that a scaling invariant Sobolev inequality implies the mean value property $(\mathcal{M})$, a classical fact proven by a well-known Moser iteration argument. For the sake of completeness, we will spell out the details below. We continue to assume that M admits a proper Lipschitz function $\rho > 0$ satisfying (1.7), namely,

$$\frac{1}{2} \leq |\nabla \rho| \leq 1 \text{ and } \Delta \rho \leq \frac{m}{\rho} \tag{6.1}$$

in the weak sense for $\rho \geq R_0$. The sublevel and level sets of ρ are denoted by

$$\begin{aligned}D\,(r) &= \{x \in M : \rho(x) < r\}\\ \Sigma\,(r) &= \{x \in M : \rho(x) = r\}\,,\end{aligned}$$

respectively, and their volume and area by

$$\begin{aligned}\mathrm{V}(r) &= \mathrm{Vol}(D(r))\\ \mathrm{A}(r) &= \mathrm{Area}(\Sigma(r)).\end{aligned}$$

Recall that (M,g) satisfies the Sobolev inequality $(\mathcal{S})$ if there exist constants $\mu>1$ and $A>0$ such that

$$\left(\fint_{D(R)}\phi^{2\mu}\right)^{\frac{1}{\mu}}\leq AR^2\fint_{D(R)}\left(\left|\nabla\phi\right|^2+\sigma\phi^2\right) \tag{6.2}$$

for $\phi\in C_0^\infty(D(R))$ and $R\geq R_0$. Here and in the following,

$$\fint_\Omega u=\frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega u$$

for a compact subset $\Omega\subset M$ and an integrable function u on Ω. We denote ν to be the number determined by

$$\frac{1}{\mu}+\frac{1}{\nu}=1.$$

Proposition 6.1. *Assume that (M,g) admits a proper function ρ satisfying (6.1) and that the Sobolev inequality $(\mathcal{S})$ holds. Then there exists a constant $C(A,\mu)>0$ such that*

$$\sup_{\Sigma(R)}u\leq\frac{C(A,\mu)}{\theta^{2\nu}\mathrm{V}(2R)}\int_{D((1+\theta)R)\setminus D\left(\frac{R}{4}\right)}u$$

for any $0<\theta\leq1$ and a positive subsolution u of $\Delta u\geq\sigma u$ on $D(2R)\setminus D(R_0)$ with $R\geq 4R_0$. In particular, M has the mean value property $(\mathcal{M})$.

Proof. The proof is by Moser iteration and can be found in Chapter 19 of [22]. We may assume $0<\theta<\frac{1}{8}$. For a function ϕ with compact support in $D(2R)$ and a positive integer $k\geq1$, applying the Sobolev inequality (6.2) to ϕu^k, we get

$$\left(\int_{D(2R)}\left(u^k\phi\right)^{2\mu}\right)^{\frac{1}{\mu}}\leq\frac{4AR^2}{\mathrm{V}(2R)^{\frac{1}{\nu}}}\int_{D(2R)}\left(\left|\nabla\left(u^k\phi\right)\right|^2+\sigma u^{2k}\phi^2\right), \tag{6.3}$$

where $\frac{1}{\nu}=1-\frac{1}{\mu}$. Integrating by parts and using $\Delta u\geq\sigma u$, we compute the first term of the right side as

$$\begin{aligned}\int_{D(2R)}\left|\nabla\left(u^k\phi\right)\right|^2&=k^2\int_{D(2R)}\left|\nabla u\right|^2u^{2k-2}\phi^2+\int_{D(2R)}\left|\nabla\phi\right|^2u^{2k}\\&\quad+\frac{1}{2}\int_{D(2R)}\left\langle\nabla u^{2k},\nabla\phi^2\right\rangle\\&=-k\left(k-1\right)\int_{D(2R)}\left|\nabla u\right|^2u^{2k-2}\phi^2-k\int_{D(2R)}\left(\Delta u\right)u^{2k-1}\phi^2\\&\quad+\int_{D(2R)}\left|\nabla\phi\right|^2u^{2k}\\&\leq-\int_{D(2R)}\sigma u^{2k}\phi^2+\int_{D(2R)}\left|\nabla\phi\right|^2u^{2k}.\end{aligned}$$

Plugging into (6.3) we conclude

$$\left(\int_{D(2R)}\left(u^k\phi\right)^{2\mu}\right)^{\frac{1}{\mu}}\leq\frac{4AR^2}{\mathrm{V}(2R)^{\frac{1}{\nu}}}\int_{D(2R)}u^{2k}\left|\nabla\phi\right|^2. \tag{6.4}$$

For fixed constants T_1, T_2, δ_1 and δ_2 with $\frac{R}{2} < T_1 < T_2 < \frac{3R}{2}$ and $0 < \delta_1,\ \delta_2 < \frac{1}{4}R$, let

$$\phi(x) = \begin{cases} 1 & \text{on } D(T_2)\backslash D(T_1) \\ \frac{1}{\delta_2}(T_2 + \delta_2 - \rho(x)) & \text{on } D(T_2+\delta_2)\backslash D(T_2) \\ \frac{1}{\delta_1}(\rho(x) - T_1 + \delta_1) & \text{on } D(T_1)\backslash D(T_1-\delta_1) \\ 0 & \text{otherwise.} \end{cases}$$

Plugging into (6.4) we get

$$\|u\|_{2k\mu,T_1,T_2} \leq \left(\frac{4AR^2}{\mathrm{V}\,(2R)^{\frac{1}{\nu}} \min\{\delta_1,\delta_2\}^2}\right)^{\frac{1}{2k}} \|u\|_{2k,T_1-\delta_1,T_2+\delta_2}\,, \tag{6.5}$$

where

$$\|u\|_{a,T_1,T_2} = \left(\int_{D(T_2)\backslash D(T_1)} u^a\right)^{\frac{1}{a}}.$$

We now iterate the inequality. Fix $\frac{3R}{8} < R_1 < R_2 < \frac{5}{4}R$ and $0 < \epsilon_1,\ \epsilon_2 < \frac{1}{8}$. For each integer $i \geq 0$, set

$$\begin{aligned} k_i &= \mu^i \\ \delta_{1,i} &= \frac{\epsilon_1 R_1}{2^{i+1}}, \quad \delta_{2,i} = \frac{\epsilon_2 R_2}{2^{i+1}} \\ T_{1,i} &= (1-\epsilon_1)\,R_1 + \sum_{j=0}^{i} \delta_{1,j}, \quad T_{2,i} = (1+\epsilon_2)\,R_2 - \sum_{j=0}^{i} \delta_{2,j}. \end{aligned}$$

Applying (6.5) with $k = k_j$, $\delta_1 = \delta_{1,j}$, $\delta_2 = \delta_{2,j}$ and $T_1 = T_{1,j}$ and $T_2 = T_{2,j}$, and iterating from $j = 0$ to $j = i$, one obtains

$$\|u\|_{2\mu^{i+1},T_{1,i},T_{2,i}} \leq \prod_{j=0}^{i} \left(\frac{4AR^2}{\mathrm{V}(2R)^{\frac{1}{\nu}} \min\{\delta_{1,j},\delta_{2,j}\}^2}\right)^{\frac{1}{2\mu^j}} \|u\|_{2,(1-\epsilon_1)R_1,(1+\epsilon_2)R_2}\,.$$

Letting $i \to \infty$ yields

$$\|u\|_{\infty,R_1,R_2} \leq \left(\frac{C(\mu)A}{\mathrm{V}(2R)^{\frac{1}{\nu}} \min\{\epsilon_1,\epsilon_2\}^2}\right)^{\frac{\nu}{2}} \|u\|_{2,(1-\epsilon_1)R_1,(1+\epsilon_2)R_2}$$

for $\frac{3R}{8} < R_1 < R_2 < \frac{5}{4}R$ and $0 < \epsilon_1,\ \epsilon_2 < \frac{1}{8}$.

So we have

$$\begin{aligned} \|u\|_{\infty,R_1,R_2} &\leq \frac{C\,(A,\mu)}{\mathrm{V}\,(2R)^{\frac{1}{2}} \min\{\epsilon_1,\epsilon_2\}^{\nu}} \|u\|_{2,(1-\epsilon_1)R_1,(1+\epsilon_2)R_2} \\ &\leq \frac{C\,(A,\mu)}{\mathrm{V}\,(2R)^{\frac{1}{2}} \min\{\epsilon_1,\epsilon_2\}^{\nu}} \|u\|^{\frac{1}{2}}_{\infty,(1-\epsilon_1)R_1,(1+\epsilon_2)R_2} \|u\|^{\frac{1}{2}}_{1,(1-\epsilon_1)R_1,(1+\epsilon_2)R_2}\,. \end{aligned} \tag{6.6}$$

Applying (6.6) for each i with

$$\begin{aligned} R_1 = R_{1,i} &= \frac{R}{2} - \frac{\theta R}{2}\sum_{j=1}^{i}\frac{1}{2^j}, \qquad \epsilon_1 = \epsilon_{1,i} = 1 - \frac{R_{1,i+1}}{R_{1,i}} \\ R_2 = R_{2,i} &= R + \theta R \sum_{j=1}^{i}\frac{1}{2^j}, \qquad \epsilon_2 = \epsilon_{2,i} = \frac{R_{2,i+1}}{R_{2,i}} - 1 \end{aligned}$$

and iterating, we conclude that

$$\|u\|_{\infty,\frac{R}{2},R} \leq \frac{C(A,\mu)}{\mathrm{V}(2R)\theta^{2\nu}} \|u\|_{1,(1-\theta)\frac{R}{2},(1+\theta)R}.$$

This proves the result. □

We note that only $|\nabla \rho| \leq 1$ on $M \setminus D(R_0)$ from (6.1) was used in the proof of Proposition 6.1. The following corollary is immediate.

Corollary 6.2. *Assume that (M, g) admits a proper function ρ satisfying (6.1) and that the Sobolev inequality $(\mathcal{S})$ holds. Then there exists $C(A, \mu) > 0$ such that*

$$\sup_{D(R)} u \leq \frac{C\,(A,\mu)}{\theta^{2\nu}} \fint_{D((1+\theta)R)} u$$

for any $0 < \theta \leq 1$ and positive subsolution u of $\Delta u \geq \sigma u$ on $D(2R)$ with $R \geq R_0$.

By combining Proposition 6.1 with Theorem 5.2, we have the following result.

Theorem 6.3. *Assume that (M, g) admits a proper function ρ satisfying (6.1) and that the Sobolev inequality $(\mathcal{S})$ holds. Suppose that σ decays quadratically. Then $\dim \mathcal{P} \leq \Gamma(m, A, \nu, \alpha)$ provided that $\alpha < \infty$ for some $q > \nu - \frac{1}{2}$. In the case $q = \nu - \frac{1}{2}$, the same conclusion holds for some $\Gamma(m, A, \nu)$ when $\alpha \leq \alpha_0(m, A, \nu)$, a sufficiently small positive constant. Consequently, the number of ends $e(M)$ of M satisfies the same estimate as well.*

We also remark that Proposition 6.1 can be localized to an end E of M as follows. For $r \geq R_0$, we denote

$$\begin{aligned} E(r) &= E \cap D(r), \\ \partial E(r) &= E \cap \Sigma(r). \end{aligned}$$

Corollary 6.4. *Assume that (M, g) admits a proper function ρ satisfying (6.1) and that the Sobolev inequality $(\mathcal{S})$ holds. Then there exists a constant $C(A, \mu) > 0$ such that*

$$\sup_{\partial E(R)} u \leq \frac{C(A,\mu)}{\theta^{2\nu}\mathrm{V}(2R)} \int_{E((1+\theta)R)\setminus E\left(\frac{R}{4}\right)} u$$

for any $0 < \theta \leq 1$ and positive subsolution u of $\Delta u \geq \sigma u$ on $E(2R) \backslash E(R_0)$ with $R \geq 4R_0$.

Proof. In the proof of Proposition 6.1 one may choose the cut-off ϕ with support in the end E as follows.

$$\phi(x) = \begin{cases} 1 & \text{on } E(T_2)\backslash D(T_1) \\ \frac{1}{\delta_2}(T_2 + \delta_2 - \rho(x)) & \text{on } E(T_2+\delta_2)\backslash D(T_2) \\ \frac{1}{\delta_1}(\rho(x) - T_1 + \delta_1) & \text{on } E(T_1)\backslash D(T_1 - \delta_1) \\ 0 & \text{otherwise.} \end{cases}$$

with $\frac{R}{2} < T_1 < T_2 < \frac{3R}{2}$ and $0 < \delta_1,\ \delta_2 < \frac{1}{4}R$. The rest of the proof is verbatim. □

It is perhaps worth pointing out that the normalization in Corollary 6.4 is by the volume of $D(2R)$, not of its intersection with E. We now apply this localized version to improve Corollary 4.6.

For an end E of M, define

$$\alpha_E = \limsup_{R\to\infty} \left(\frac{R^{2q}}{\mathrm{A}(R)} \int_{\partial E(R)} \sigma^q \right)^{\frac{1}{q}}. \tag{6.7}$$

Corollary 6.5. *Assume that (M,g) admits a proper function ρ satisfying (6.1) and that the Sobolev inequality $(\mathcal{S})$ holds. Suppose that σ decays quadratically along E. Then there exists $\Gamma(m,A,\nu,\alpha_E) > 0$ such that*

$$u \le \Lambda \left(\rho^{\Gamma} + 1\right) \quad \text{on } E$$

for any positive solution u to $\Delta u = \sigma u$ on E, where $\Lambda > 0$ is a constant depending on u, provided that $\alpha_E < \infty$ for some $q > \nu - \frac{1}{2}$. In the case $q = \nu - \frac{1}{2}$, the same conclusion holds for some $\Gamma(m,A,\nu) > 0$ when $\alpha_E \le \alpha_0(m,A,\nu)$, a sufficiently small positive constant.

Proof. First, Lemma 4.3 can be localized to the end E to yield

$$\frac{d}{dr}\left(\frac{1}{r^{4m}} \int_{\partial E(r)} u\,|\nabla \rho|\right) \le \frac{1}{r^{4m}} \int_{E(r)\setminus E(r_0)} \sigma u + \frac{1}{r^{4m}} \int_{\partial E(r_0)} \frac{\langle \nabla u, \nabla \rho\rangle}{|\nabla \rho|}$$

for any $r_0 \ge R_0$. Using the fact that σ decays quadratically along E, one concludes that u is of polynomial growth along E by adopting the same argument as in Proposition 4.4.

Recall by Corollary 6.4 that

$$\sup_{\partial E(R)} u \le \frac{C(A,\mu)}{\theta^{2\nu}} \frac{1}{\mathrm{V}(2R)} \int_{E((1+\theta)R)\setminus E(R_0)} u \tag{6.8}$$

for $R > 4R_0$ and $0 < \theta \le 1$. Following the proof of (4.25) we obtain that the function

$$\chi_E(r) = \int_{E(r)\setminus E(R_0)} u \frac{|\nabla \rho|^2}{\rho^{4m}}$$

satisfies the following inequality:

$$r^{4m}\chi_E''(r) \le \frac{C_0 \bar{\alpha}_E}{\theta^{\frac{2\nu}{q}}} \int_{r_0}^{r} \chi_E((1+\theta)t)^{\frac{1}{q}} (\chi_E'(t))^{1-\frac{1}{q}} t^{4m-2-\frac{1}{q}}\, dt + \Lambda_0$$

for $r \ge r_0$ and $0 < \theta \le 1$, where

$$\Lambda_0 = \int_{\partial E(r_0)} (u + |\nabla u|)$$

and $\bar{\alpha}_E = \min\{\alpha_E, 1\}$, with the constant C_0 depending only on m, A, μ and α_E.

Using an induction argument as in Theorem 4.5, we arrive at

$$\int_{\partial E(r)} u \le \Lambda r^{C(m,A,\mu,\alpha_E)}$$

for $r \ge r_0$. Integrating in r and using (6.8), we conclude

$$u \le \widetilde{\Lambda}\left(\rho^{\Gamma_\epsilon} + 1\right)$$

on end E. This proves the result. $\square$

Corresponding to an end E, let u_E be the positive solution of $\Delta u_E = \sigma u_E$ on M constructed in Theorem 3.2. Then $0 < u_E \leq 1$ on $M \setminus E$. In particular, under the assumptions of Corollary 6.5, u_E must be of polynomial growth on M with the given growth order.

Acknowledgment. The first author was partially supported by NSF grant DMS-1506220 and by a Leverhulme Trust Visiting Professorship VP2-2018-029. The second author was supported by a Leverhulme Trust Research Project Grant RPG-2016-174.

References

[1] S. Agmon, Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-Body Schrödinger operators, Mathematical Notes, vol. 29, Princeton University Press, Princeton, NJ, 1982.

[2] W. K. Allard, First variation of a varifold, Ann. Math. 95 (1972), 417-492.

[3] H. D. Cao, B. L. Chen and X. P. Zhu, Recent developments on Hamilton's Ricci flow. Surveys in differential geometry. Vol. XII. Geometric flows, 47-112, Surv. Differ. Geom., 12, Int. Press, Somerville, MA, 2008.

[4] H. D. Cao and D. Zhou, On complete gradient shrinking Ricci solitons, J. Differential Geom. 85 (2010), 175-186.

[5] H.-D. Cao, Y. Shen, S. Zhu, The structure of stable minimal hypersurfaces in $\mathbb{R}^{m+1}$, Math. Res. Lett. 4 (5) (1997), 637–644.

[6] X. Cao and Q. Zhang, The conjugate heat equation and ancient solutions of the Ricci flow, Adv. Math. 228 (5) (2011), 2891-2919.

[7] J. Carron, L2-Cohomologie et inégalités de Sobolev, Math. Ann. 314 (1999), 614–639.

[8] B.L. Chen, Strong uniqueness of the Ricci flow, J. Differential Geom. 82 (2009), no. 2, 362-382.

[9] S. Y. Cheng and S.T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28 (1975), 333-354.
c

[10] T.H. Colding and W.P. Minicozzi II, Harmonic functions on manifolds, Ann. of Math. 146 (1997), 725-747.

[11] T.H. Colding and W.P. Minicozzi II, Harmonic functions with polynomial growth, J. Differential Geom. 46 (1997), 1-77.

[12] R. J. Conlon, A. Deruelle and S. Sun, Classification results for expanding and shrinking gradient Kähler-Ricci solitons, arXiv:1904.00147v2.

[13] J. Enders, R. Müller, P. Topping, On Type-I singularities in Ricci flow, Comm. Anal. Geom. 19 (2011), no. 5, 905-922.

[14] F. Fang, J. Man and Z. Zhang, Complete gradient shrinking Ricci solitons have finite topological type, C. R. Math. Acad. Sci. Paris 346 (2008), no. 11-12, 653-656.

[15] D. Hoffman and J. Spruck, Sobolev and isoperimetric inequalities for Riemannian submanifolds, Comm. Pure Appl. Math. 27 (1974), 715–727.

[16] E. Hebey, Sobolev Spaces on Riemannian Manifolds, Lecture Notes in Mathematics, vol. 1635, Springer, Berlin, 1996.

[17] R. Haslhofer and R. Müller, A compactness theorem for complete Ricci shrinkers, Geom. Funct. Anal. 21 (2011), 1091-1116.

[18] R. Hamilton, The formation of singularities in the Ricci flow, Surveys in Differential Geom. 2 (1995), 7-136, International Press.

[19] G. Huisken, Flow by the mean curvature of convex surfaces into spheres, J. Differential Geom. 20 (1984), 237–266.

[20] N. Kapouleas, S. Kleene and N. Möller, Mean curvature self-shrinkers of high genus: non-compact examples. J. Reine Angew. Math. 739 (2018), 1-39.

[21] P. Li, Harmonic sections of polynomial growth. Math. Res. Lett. 4 (1997), 35-44.

[22] P. Li, Geometric Analysis, Cambridge Studies in Advanced Mathematics, **134**, Cambridge University Press, Cambridge, 2012.

[23] P. Li and L.F. Tam, Harmonic functions and the structure of complete manifolds, J. Differential Geom. 35 (1992), 359-383.
[24] P. Li and J. Wang, Complete manifolds with positive spectrum, J. Differential Geom. 58 (2001), 501–534.
[25] P. Li and J. Wang, Weighted Poincaré inequality and rigidity of complete manifolds, Ann. Sci. Ecole Norm. Sup. 39 (2006), 921–982.
[26] Y. Li and B. Wang, Heat kernel on Ricci shrinkers, arXiv:1901.05691.
[27] J. H. Michael and L. M. Simon, Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$, Comm. Pure Appl. Math. 26 (1973), 361-379.
[28] O. Munteanu and J. Wang, Topology of Kähler Ricci solitons, J. Differential Geom. 100 (2015), 109-128.
[29] O. Munteanu and J. Wang, Geometry of shrinking Ricci solitons, Compos. Math. 151(2015), 2273-2300.
[30] O. Munteanu and J. Wang, Structure at infinity for shrinking Ricci solitons, Ann. Sci. École Norm. Sup. 52 (2019), 891-925.
[31] A. Naber, Noncompact Shrinking 4-Solitons with Nonnegative Curvature, J. Reine Angew. Math. 645 (2010), 125-153.
[32] L. Ni, Gap theorems for minimal submanifolds in $\mathbb{R}^{n+1}$, Comm. Anal. Geom. 9 (3) (2001), 641–656.
[33] L. Ni and N. Wallach, On a classification of gradient shrinking solitons, Math. Res. Lett. 15 (2008), no. 5, 941–955.
[34] X. H. Nguyen, Construction of complete embedded self-similar surfaces under mean curvature flow, Part III. Duke Math. J. 163 (2014), no. 11, 2023-2056.
[35] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math. DG/0211159.
[36] S. Pigola, M. Rimoldi and A. Setti, Vanishing theorems on Riemannian manifolds, and geometric applications, J. Funct. Anal. 229 (2005), 424 – 461.
[37] L. Saloff-Coste, Aspects of Sobolev-type inequalities, Cambridge Univ. Press, 2002.
[38] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, J. Differential. Geom. 36 (1992), 417-450.
[39] R. Schoen and S.T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. Math. 92 (1988), 47-71.
[40] N. Sesum, Convergence of a Ricci flow towards a Ricci soliton, Comm. Anal. Geom. 14 (2006), no.2, 283-343.
[41] A. Sun and Z. Wang, Compactness of self-shrinkers in $\mathbb{R}^3$ with fixed genus, Adv. Math. https://doi.org/10.1016/j.aim.2020.107110
[42] P. Topping, Diameter control under Ricci flow, Comm. Anal. Geom. 13 (2005), 1039-1055.
[43] P. Topping, Relating diameter and mean curvature for submanifolds of Euclidean space, Comment. Math. Helv. 83 (2008), 539-546.
[44] L. Wang, Uniqueness of self-similar shrinkers with asymptotically conical ends, J. Amer. Math. Soc. 27 (2014), no. 3, 613-638.

Department of Mathematics, University of Connecticut, Storrs, CT 06268, USA
Email address: `ovidiu.munteanu@uconn.edu`

Department of Mathematics, University of Warwick, Coventry CV7 4AL, UK
Email address: `felix.schulze@warwick.ac.uk`

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
Email address: `jiaping@math.umn.edu`